

The coupling method for inhomogeneous random intersection graphs.

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Abstract

We present new results concerning threshold functions for a wide family of random intersection graphs. To this end we apply the coupling method used for establishing threshold functions for homogeneous random intersection graphs introduced by Karoński, Scheinerman, and Singer–Cohen. In the case of inhomogeneous random intersection graphs the method has to be considerably modified and extended. By means of the altered method we are able to establish threshold functions for a general random intersection graph for such properties as k -connectivity, matching containment or hamiltonicity. Moreover using the new approach we manage to sharpen the best known results concerning homogeneous random intersection graph.

1 Introduction

The first random intersection graph model was introduced by Karoński, Scheinerman, and Singer–Cohen [11]. Since then it has been attracting attention mainly because of its wide applications, for example: "gate matrix layout" for VLSI design (see e.g. [11]), cluster analysis and classification (see e.g. [9]), analysis of complex networks (see e.g. [7, ?]), secure wireless networks (see e.g. [2]) and epidemics ([6]). Several generalisations of the model has been proposed, mainly in order to adapt it to use in some particular purpose. In this paper we consider the $\mathcal{G}(n, m, \bar{p})$ model studied for example in [1, 5, 12]. Alternative ways of generalizing the model defined in [11] are given for example in [7] and [9].

In a random intersection graph $\mathcal{G}(n, m, \bar{p})$ there is a set of n vertices $\mathcal{V} = \{v_1, \dots, v_n\}$, an auxiliary set of $m = m(n)$ features $\mathcal{W} = \{w_1, \dots, w_{m(n)}\}$, and a vector $\bar{p}(n) = (p_1, \dots, p_{m(n)})$ such that $p_i \in (0, 1)$, for each $1 \leq i \leq m$. Each vertex $v \in \mathcal{V}$ adds a feature $w_i \in \mathcal{W}$ to its feature set $W(v)$ with probability p_i independently of all other properties and features. Any two vertices $v, v' \in \mathcal{V}$ are connected by an edge in $\mathcal{G}(n, m, \bar{p})$ if $W(v)$ and $W(v')$ intersect. If $\bar{p}(n) = (p, \dots, p)$ for some $p \in (0, 1)$ then $\mathcal{G}(n, m, \bar{p})$ is a random intersection graph defined in [11]. We denote it by $\mathcal{G}(n, m, p)$.

The random intersection graph model is very flexible and its properties change a lot if we alter the parameters. For example $\mathcal{G}(n, m, p)$ for some ranges of parameters behaves

similarly to a random graph with independent edges (see [8, 13]) but in some cases it exhibit large dependencies between edge appearance (see for example [11, 15]). It was proved in [14] that in both cases $\mathcal{G}(n, m, p)$ may be coupled with a random graph with independent edges so that with probability tending to 1 as $n \rightarrow \infty$, $\mathcal{G}(n, m, p)$ is an overgraph of a graph with independent edges. It is also explained how this coupling may be used to obtain sharp results on threshold functions for $\mathcal{G}(n, m, p)$. Such properties as connectivity, a Hamilton cycle containment or a matching containment are given as examples. In general, the coupling technique provides a very elegant method to get bounds on threshold functions for random intersection graphs for a large class of properties.

However the proof presented in [14] cannot be straightforward generalised to $\mathcal{G}(n, m, \bar{p})$ with arbitrary $\bar{p}(n)$. First of all it differentiates between cases $np \rightarrow 0$ and $np \rightarrow \infty$. Moreover the method does not give sharp results for np tending to a constant. In this article we modify and extend the techniques used in [14] in order to overcome these constraints. First of all, to get the general result, we couple $\mathcal{G}(n, m, \bar{p})$ with an auxiliary random graph which does not have fully independent edges. Therefore we need to prove some additional facts about the auxiliary random graph. Moreover we need sharp bounds on the minimum degree threshold function for $\mathcal{G}(n, m, \bar{p})$. Due to edge dependencies, estimation of moments of the random variable counting vertices with a given degree in $\mathcal{G}(n, m, \bar{p})$ is complicated. Therefore we suggest a different approach to resolve the problem. We divide $\mathcal{G}(n, m, \bar{p})$ into subgraphs so that the solution of a coupon collector problem combined with the method of moments provide the answer. This new approach to the coupling method allow us to obtain better results on threshold functions for $\mathcal{G}(n, m, p)$ and by this means resolve open problems left over in [14].

Concluding, we provide a general method to establish bounds on threshold functions for many properties for $\mathcal{G}(n, m, \bar{p})$. By means of the method we are able to obtain sharp thresholds for k -connectivity, perfect matching containment and hamiltonicity for the general model. Last but not least we considerably improve known results concerning $\mathcal{G}(n, m, p)$.

All limits in the paper are taken as $n \rightarrow \infty$. Throughout the paper we use standard asymptotic notation $o(\cdot)$, $O(\cdot)$, $\Omega(\cdot)$, $\Theta(\cdot)$, \sim , \ll , and \gg defined as in [10]. By $\text{Bin}(n, p)$ and $\text{Po}(\lambda)$ we denote the binomial distribution with parameters n, p and the Poisson distribution with expected value λ , respectively. We also use the phrase “with high probability” to say with probability tending to one as n tends to infinity. All inequalities hold for n large enough.

2 Main Results

In the article we compare random intersection graph $\mathcal{G}(n, m, \bar{p})$ with a sum of a random graph with independent edges $G_2(n, \hat{p}_2)$ and a random graph $G_3(n, \hat{p}_3)$ constructed on the base of a random 3-uniform hypergraph with independent hyperedges. Generally, for any $\hat{p} = \hat{p}(n) \in [0; 1]$ and $i = 2, \dots, n$, let $H_i(n, \hat{p})$ be an i -uniform hypergraph with the vertex set \mathcal{V} in which each i -element subset of \mathcal{V} is added to the hyperedge set independently with probability \hat{p} . $G_i(n, \hat{p})$ is a graph with the vertex set \mathcal{V} and an edge set consisting of those two element subsets of \mathcal{V} which are subsets of at least one hyperedge of $H_i(n, \hat{p})$.

We consider monotone graph properties of random graphs. For the family \mathcal{G} of all graphs with the vertex set \mathcal{V} , we call $\mathcal{A} \subseteq \mathcal{G}$ a property if it is closed under isomorphism. Moreover \mathcal{A} is increasing if $G \in \mathcal{A}$ implies $G' \in \mathcal{A}$ for all $G' \in \mathcal{G}$ such that $E(G) \subseteq E(G')$ and decreasing if $\mathcal{G} \setminus \mathcal{A}$ is increasing. Increasing properties are for example: k -connectivity, containing a perfect matching and containing a Hamilton cycle.

Let $\bar{p} = (p_1, \dots, p_m)$ be such that $p_i \in (0, 1)$, for all $1 \leq i \leq m$. Define

$$(1) \quad \begin{aligned} S_1 &= \sum_{i=1}^m np_i (1 - (1 - p_i)^{n-1}); \\ S_2 &= \sum_{i=1}^m np_i \left(1 - \frac{1 - (1 - 2p_i)^n}{2np_i} \right); \\ S_3 &= \sum_{i=1}^m np_i \left(\frac{1 - (1 - 2p_i)^n}{2np_i} - (1 - p_i)^{n-1} \right); \\ S_{1,t} &= \sum_{i=1}^m t \binom{n}{t} p_i^t (1 - p_i)^{n-t}, \text{ for } t = 2, 3, \dots, n. \end{aligned}$$

The following theorem is an extension of the result obtained in [14].

Theorem 1. *Let S_1 , S_2 and S_3 be given by (1). For some function ω tending to infinity let*

$$(2) \quad \begin{aligned} \hat{p} &= \frac{S_2 - \omega \sqrt{S_2} - 2S_2^2 n^{-2}}{2 \binom{n}{2}}; \\ \hat{p}_2 &= \begin{cases} \frac{S_1 - 3S_3 - \omega \sqrt{S_1} - 2S_1^2 n^{-2}}{2 \binom{n}{2}}, & \text{for } S_3 \gg \sqrt{S_1} \text{ and } \omega^2 \ll S_3 / \sqrt{S_1}; \\ \frac{S_1 - \omega \sqrt{S_1} - 2S_1^2 n^{-2}}{2 \binom{n}{2}}, & \text{for } S_3 = O(\sqrt{S_1}); \end{cases} \\ \hat{p}_3 &= \begin{cases} \frac{S_3 - \omega \sqrt{S_1} - 6S_3^2 n^{-3}}{\binom{n}{3}}, & \text{for } S_3 \gg \sqrt{S_1} \text{ and } \omega^2 \ll S_3 / \sqrt{S_1}; \\ 0, & \text{for } S_3 = O(\sqrt{S_1}). \end{cases} \end{aligned}$$

If $S_1 \rightarrow \infty$ and $S_1 = o(n^2)$ then for any increasing property \mathcal{A} .

$$(3) \quad \liminf_{n \rightarrow \infty} \Pr \{G_2(n, \hat{p}) \in \mathcal{A}\} \leq \limsup_{n \rightarrow \infty} \Pr \{\mathcal{G}(n, m, \bar{p}) \in \mathcal{A}\},$$

$$(4) \quad \liminf_{n \rightarrow \infty} \Pr \{G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3) \in \mathcal{A}\} \leq \limsup_{n \rightarrow \infty} \Pr \{\mathcal{G}(n, m, \bar{p}) \in \mathcal{A}\}.$$

Remark 1. Assumption $S_1 \rightarrow \infty$ is natural since for $S_1 = o(1)$ with high probability $\mathcal{G}(n, m, \bar{p})$ is an empty graph.

Remark 2. S_3 is the expected number of edges in $G_3(n, \hat{p}_3)$. If $S_3 = O(\sqrt{S_1})$ then by Markov's inequality with high probability the number of edges in $G_3(n, \hat{p}_3)$ is at most $\omega \sqrt{S_1}$. Thus $S_2 = S_1 - S_3 = S_1 + O(\omega \sqrt{S_1})$ and the bound provided by (3) is as good as the one taking into consideration the edges from $G_3(n, \hat{p}_3)$.

Remark 3. Theorem is also valid for $S_1 = \Omega(n^2)$ but with

$$\begin{aligned}\hat{p} &= 1 - \exp\left(-\frac{S_2 - \omega\sqrt{S_2}}{2\binom{n}{2}}\right); \\ \hat{p}_2 &= \begin{cases} 1 - \exp\left(-\frac{S_1 - 3S_3 - \omega\sqrt{S_1}}{2\binom{n}{2}}\right), & \text{for } S_3 \gg \sqrt{S_1} \text{ and } \omega \ll S_3/\sqrt{S_1}; \\ 1 - \exp\left(-\frac{S_1 - \omega\sqrt{S_1}}{2\binom{n}{2}}\right), & \text{for } S_3 = O(\sqrt{S_1}); \end{cases} \\ \hat{p}_3 &= \begin{cases} 1 - \exp\left(-\frac{S_3 - \omega\sqrt{S_1}}{\binom{n}{3}}\right), & \text{for } S_3 \gg \sqrt{S_1} \text{ and } \omega \ll S_3/\sqrt{S_1}; \\ 0, & \text{for } S_3 = O(\sqrt{S_1}). \end{cases}\end{aligned}$$

Denote by \mathcal{C}_k , \mathcal{PM} and \mathcal{HC} the following graph properties: a graph is k -connected, has a perfect matching and has a Hamilton cycle, respectively. We will use Theorem 1 to establish threshold functions for \mathcal{C}_k , \mathcal{PM} and \mathcal{HC} in $\mathcal{G}(n, m, \bar{p})$. By \mathcal{C}_k we denote here vertex connectivity. From the proof it follows that the threshold function for edge connectivity is the same as this for \mathcal{C}_k .

For any sequence c_n with limit we write

$$(5) \quad f(c_n) = \begin{cases} 0 & \text{for } c_n \rightarrow -\infty; \\ e^{-e^{-c}} & \text{for } c_n \rightarrow c \in (-\infty; \infty); \\ 1 & \text{for } c_n \rightarrow \infty. \end{cases}$$

Theorem 2. Let $\max_{1 \leq i \leq m} p_i = o((\ln n)^{-1})$ and S_1 and $S_{1,2}$ be given by (1).

(i) If $S_1 = n(\ln n + c_n)$, then

$$\lim_{n \rightarrow \infty} \Pr \{ \mathcal{G}(n, m, \bar{p}) \in \mathcal{C}_1 \} = f(c_n),$$

where $f(c_n)$ is given by (5).

(ii) Let k be a positive integer and $a_n = \frac{S_{1,2}}{S_1}$. If $S_1 = n(\ln n + (k-1) \ln \ln n + c_n)$, then

$$\lim_{n \rightarrow \infty} \Pr \{ \mathcal{G}(n, m, \bar{p}) \in \mathcal{C}_k \} = \begin{cases} 0 & \text{for } c_n \rightarrow -\infty \text{ and } a_n \rightarrow a \in (0; 1]; \\ 1 & \text{for } c_n \rightarrow \infty. \end{cases}$$

Assumption $\max_{1 \leq i \leq m} p_i = o((\ln n)^{-1})$ is necessary to avoid awkward cases. The problem is explained in more detail in Section 4. A straightforward corollary of the above theorem is that for $S_1 = n(\ln n + c_n)$, $c_n \rightarrow -\infty$ and any $k = 1, 2, \dots, n$.

$$\lim_{n \rightarrow \infty} \Pr \{ \mathcal{G}(n, m, \bar{p}) \in \mathcal{C}_k \} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \Pr \{ \mathcal{G}(n, m, \bar{p}) \in \mathcal{HC} \} = 0.$$

Theorem 3. Let $\max_{1 \leq i \leq m} p_i = o((\ln n)^{-1})$ and S_1 be given by (1). If $S_1 = n(\ln n + c_n)$ then

$$\lim_{n \rightarrow \infty} \Pr \{ \mathcal{G}(2n, m, \bar{p}(2n)) \in \mathcal{PM} \} = f(c_{2n}),$$

where $f(\cdot)$ is given by (5).

Theorem 4. Let $\max_{1 \leq i \leq m} p_i = o((\ln n)^{-1})$, S_1 and $S_{1,2}$ be given by (1) and $a_n = \frac{S_{1,2}}{S_1}$. If $S_1 = n(\ln n + \ln \ln n + c_n)$, then

$$\lim_{n \rightarrow \infty} \Pr \{ \mathcal{G}(n, m, \bar{p}) \in \mathcal{HC} \} = \begin{cases} 0 & \text{for } c_n \rightarrow -\infty \text{ and } a_n \rightarrow a \in (0; 1]; \\ 1 & \text{for } c_n \rightarrow \infty. \end{cases}$$

Already simple corollaries of Theorems 2–4 give sharp threshold functions for $\mathcal{G}(n, m, p)$. For example.

Corollary 1. Let $m \gg \ln^2 n$ and $p(1 - (1 - p)^{n-1}) = (\ln n + c_n)/m$. Then

$$\lim_{n \rightarrow \infty} \Pr \{ \mathcal{G}(n, m, p) \in \mathcal{C}_1 \} = f(c_n)$$

and

$$\lim_{n \rightarrow \infty} \Pr \{ \mathcal{G}(2n, m, p) \in \mathcal{PM} \} = f(c_{2n}),$$

where $f(\cdot)$ is given by (5).

In particular we may state the following extension of the result from [17].

Corollary 2. Let b_n be a sequence, β and γ be constants such that $\beta\gamma(1 - e^{-\gamma}) = 1$. If

$$m = \beta n \ln n \quad \text{and} \quad p = \frac{\gamma}{n} \left(1 + \frac{b_n}{\ln n} \right) \quad \text{then}$$

$$\lim_{n \rightarrow \infty} \Pr \{ \mathcal{G}(n, m, p) \in \mathcal{C}_1 \} = f \left(\left(1 + \frac{e^{-\gamma}\gamma}{1 - e^{-\gamma}} \right) b_n \right)$$

and

$$\lim_{n \rightarrow \infty} \Pr \{ \mathcal{G}(2n, m, p(2n)) \in \mathcal{PM} \} = f \left(\left(1 + \frac{e^{-\gamma}\gamma}{1 - e^{-\gamma}} \right) b_{2n} \right),$$

where $f(\cdot)$ is given by (5).

Sometimes the method of the proof enable to improve the best known results concerning $\mathcal{G}(n, m, p)$ even more.

Theorem 5. Let $m \gg \ln^2 n$ and

$$(6) \quad p(1 - (1 - p)^{n-1}) = \frac{\ln n + \ln \left(\max \left\{ 1, \ln \left(\frac{npe^{-np} \ln n}{1 - e^{-np}} \right) \right\} \right) + c_n}{m}.$$

Then

$$\lim_{n \rightarrow \infty} \Pr \{ \mathcal{G}(n, m, p) \in \mathcal{HC} \} = \begin{cases} 0 & \text{for } c_n \rightarrow -\infty; \\ 1 & \text{for } c_n \rightarrow \infty. \end{cases}$$

Theorem 6. *Let $m \gg \ln^2 n$ and k be a positive integer. If*

$$p(1 - (1 - p)^{n-1}) = \frac{\ln n + \ln \left(\max \left\{ 1, (np)^{k-1} \left(\left(\frac{e^{-np} \ln n}{1 - e^{-np}} \right)^{k-1} + \frac{e^{-np} \ln n}{1 - e^{-np}} \right) \right\} \right) + c_n}{m},$$

then

$$\lim_{n \rightarrow \infty} \Pr \{ \mathcal{G}(n, m, p) \in \mathcal{C}_k \} = \begin{cases} 0 & \text{for } c_n \rightarrow -\infty; \\ 1 & \text{for } c_n \rightarrow \infty. \end{cases}$$

One of the question posed in [14] concerned the range of $m = m(n)$ for which the threshold function for \mathcal{C}_k for $\mathcal{G}(n, m, p)$ coincides with this for $\delta(\mathcal{G}(n, m, p)) \geq 1$. Moreover we may ask when threshold function for \mathcal{C}_k for $\mathcal{G}(n, m, p)$ is the same as this for \mathcal{C}_k for $G_2(n, \hat{p})$ with $\hat{p} = mp^2$. Theorem 6 gives a final answer to these questions.

Corollary 3. *Let k be a positive integer. If*

$$p(1 - (1 - p)^{n-1}) = \begin{cases} \frac{\ln n + c_n}{m}, & \text{for } \ln^2 n \ll m \ll \frac{n \ln n}{\ln \ln n} \\ \frac{\ln n + (k-1) \ln \ln n + c_n}{m}, & \text{for } m = \Omega(n \ln n); \end{cases}$$

then

$$\lim_{n \rightarrow \infty} \Pr \{ \mathcal{G}(n, m, p) \in \mathcal{C}_k \} = \begin{cases} 0 & \text{for } c_n \rightarrow -\infty; \\ 1 & \text{for } c_n \rightarrow \infty. \end{cases}$$

The proof is divided as follows. Section 3 describes the coupling used to establish threshold functions and presents the proof of Theorem 1. In Section 4 we give minimum degree thresholds for $\mathcal{G}(n, m, \bar{p})$. Section 5 is dedicated to the properties of the auxiliary random graphs used in the coupling established in Theorem 1. The proofs of the remaining theorems are presented in Section 6.

3 Coupling

In this section we present a proof of Theorem 1. In the proof we use auxiliary random graph models $\mathbb{G}_{*i}(n, M)$, $i = 2, 3, \dots, n$, in which M is a random variable with non-negative integer values. For $i = 2, \dots, n$, $\mathbb{G}_{*i}(n, M)$ is constructed on the basis of a random hypergraph $\mathbb{H}_{*i}(n, M)$. $\mathbb{H}_{*i}(n, M)$ is a random hypergraph with the vertex set \mathcal{V} in which the hyperedge set is constructed by sampling M times with repetition elements from the set of all i -element subsets of \mathcal{V} (all sets which are chosen several times are added only once to the hyperedge set). $\mathbb{G}_{*i}(n, M)$ is a graph with the vertex set \mathcal{V} in which $v, v' \in \mathcal{V}$ are connected by an edge if $\{v, v'\}$ is contained in at least one of the hyperedges of $\mathbb{H}_{*i}(n, M)$. If M equals a constant t with probability one or has the Poisson distribution, we write $\mathbb{G}_{*i}(n, t)$ or $\mathbb{G}_{*i}(n, \text{Po}(\cdot))$, respectively. Recall that similarly $G_i(n, \hat{p})$ is constructed on the basis of $H_i(n, \hat{p})$ – a hypergraph with independent hyperedges.

In this paper we treat random graphs as random variables. By a coupling $(\mathbb{G}_1, \mathbb{G}_2)$ of two random variables \mathbb{G}_1 and \mathbb{G}_2 we mean a choice of a probability space on which a random vector $(\mathbb{G}'_1, \mathbb{G}'_2)$ is defined and \mathbb{G}'_1 and \mathbb{G}'_2 have the same distributions as \mathbb{G}_1 and

\mathbb{G}_2 , respectively. For simplicity of notation we will not differentiate between $(\mathbb{G}'_1, \mathbb{G}'_2)$ and $(\mathbb{G}_1, \mathbb{G}_2)$. For two graph valued random variables \mathbb{G}_1 and \mathbb{G}_2 we write

$$\mathbb{G}_1 \preceq \mathbb{G}_2 \quad \text{or} \quad \mathbb{G}_1 \preceq_{1-o(1)} \mathbb{G}_2,$$

if there exists a coupling $(\mathbb{G}_1, \mathbb{G}_2)$, such that in the probability space of the coupling \mathbb{G}_1 is a subgraph of \mathbb{G}_2 with probability 1 or $1 - o(1)$, respectively. Moreover, we write

$$\mathbb{G}_1 = \mathbb{G}_2,$$

if \mathbb{G}_1 and \mathbb{G}_2 have the same probability distribution (equivalently there exists a coupling $(\mathbb{G}_1, \mathbb{G}_2)$ such that $\mathbb{G}_1 = \mathbb{G}_2$ with probability one).

Note that, for any λ , in $\mathbb{H}_{*i}(n, \text{Po}(\lambda))$ each edge appears independently with probability $1 - \exp(-\lambda/\binom{n}{i})$ (see [8]). Thus

$$(7) \quad \mathbb{G}_{*i}(n, \text{Po}(\lambda)) = G_i \left(n, 1 - \exp \left(-\lambda / \binom{n}{i} \right) \right).$$

We gather here a few useful facts concerning couplings of random graphs. For proofs see [13, 14].

Fact 1. *Let M_n be a sequence of random variables and let t_n be a sequence of numbers.*

(i) *If $\Pr \{M_n \geq t_n\} = o(1)$ then $\mathbb{G}_{*i}(n, M_n) \preceq_{1-o(1)} \mathbb{G}_{*i}(n, t_n)$.*

(ii) *If $\Pr \{M_n \leq t_n\} = o(1)$ then $\mathbb{G}_{*i}(n, t_n) \preceq_{1-o(1)} \mathbb{G}_{*i}(n, M_n)$.*

Fact 2. *Let $(\mathbb{G}_i)_{i=1, \dots, m}$ and $(\mathbb{G}'_i)_{i=1, \dots, m}$ be sequences of independent random graphs. If*

$$\mathbb{G}_i \preceq \mathbb{G}'_i, \text{ for all } i = 1, \dots, m,$$

then

$$\bigcup_{i=1}^m \mathbb{G}_i \preceq \bigcup_{i=1}^m \mathbb{G}'_i.$$

Fact 3. *Let $\mathbb{G}_1 = \mathbb{G}_1(n)$, $\mathbb{G}_2 = \mathbb{G}_2(n)$, and $\mathbb{G}_3 = \mathbb{G}_3(n)$ be random graphs. If*

$$\mathbb{G}_1 \preceq_{1-o(1)} \mathbb{G}_2 \quad \text{and} \quad \mathbb{G}_2 \preceq_{1-o(1)} \mathbb{G}_3$$

then

$$\mathbb{G}_1 \preceq_{1-o(1)} \mathbb{G}_3.$$

Fact 4. *Let $\mathbb{G}_1 = \mathbb{G}_1(n)$ and $\mathbb{G}_2 = \mathbb{G}_2(n)$ be two random graphs, such that*

$$(8) \quad \mathbb{G}_1 \preceq \mathbb{G}_2 \quad \text{or} \quad \mathbb{G}_1 \preceq_{1-o(1)} \mathbb{G}_2.$$

Then for any increasing property \mathcal{A}

$$\liminf_{n \rightarrow \infty} \Pr \{ \mathbb{G}_1(n) \in \mathcal{A} \} \leq \limsup_{n \rightarrow \infty} \Pr \{ \mathbb{G}_2(n) \in \mathcal{A} \}.$$

Proof. Define event $\mathcal{E} := \{\mathbb{G}_1 \subseteq \mathbb{G}_2\}$ on the probability space of the coupling $(\mathbb{G}_1, \mathbb{G}_2)$ existing by (8). Then for any increasing property \mathcal{A}

$$\begin{aligned}
\Pr\{\mathbb{G}_2 \in \mathcal{A}\} &\geq \Pr\{\mathbb{G}_2 \in \mathcal{A} | \mathcal{E}\} \Pr\{\mathcal{E}\} \\
&\geq \Pr\{\mathbb{G}_1 \in \mathcal{A} | \mathcal{E}\} \Pr\{\mathcal{E}\} \\
&= \Pr\{\{\mathbb{G}_1 \in \mathcal{A}\} \cap \mathcal{E}\} \\
&= \Pr\{\mathbb{G}_1 \in \mathcal{A}\} + \Pr\{\mathcal{E}\} - \Pr\{\{\mathbb{G}_1 \in \mathcal{A}\} \cup \mathcal{E}\} \\
&\geq \Pr\{\mathbb{G}_1 \in \mathcal{A}\} + \Pr\{\mathcal{E}\} - 1 \\
&= \Pr\{\mathbb{G}_1 \in \mathcal{A}\} + o(1).
\end{aligned}$$

The result follows by taking $n \rightarrow \infty$ □

Proof of Theorem 1. We will show only (4) in the case $S_3 \gg \sqrt{S_1}$. The remaining cases follow by similar arguments. Here we should note that $S_2 = S_1 - S_3$ and $S_2 = \Theta(S_1)$.

Let $w_i \in \mathcal{W}$. Denote by V_i the set of vertices which have chosen feature w_i (i.e. $V_i = \{v \in \mathcal{V} : w_i \in W(v)\}$). Let

$$(9) \quad V'_i = \begin{cases} V_i & \text{for } |V_i| \geq 2; \\ \emptyset & \text{otherwise.} \end{cases}$$

For each $1 \leq i \leq m$ define

$$(10) \quad \begin{aligned} X_i &= |V_i|; \\ Y_i &= |V'_i|; \\ Z_i &= \mathbb{I}_{\{Y_i \text{ is odd}\}}, \end{aligned}$$

where \mathbb{I}_A is an indicator random variable of the event A . Note that X_i , $1 \leq i \leq m$, are independent random variables with binomial distributions $\text{Bin}(n, p_i)$, $1 \leq i \leq m$. Now let

$$M_2 = \sum_{1 \leq i \leq m} \frac{Y_i - 3Z_i}{2} \quad \text{and} \quad M_3 = \sum_{1 \leq i \leq m} Z_i.$$

Let $\mathcal{G}[V'_i]$ be a graph with the vertex set \mathcal{V} and the edge set containing those edges from $\mathcal{G}(n, m, \bar{p})$ which have both ends in V'_i (i.e. its edges form a clique with the vertex set V'_i). For each $1 \leq i \leq m$, we construct independently a coupling of $\mathbb{G}_{*2}(n, \frac{Y_i - 3Z_i}{2}) \cup \mathbb{G}_{*3}(n, Z_i)$ and $\mathcal{G}[V'_i]$. Given $Y_i = y_i$ and $Z_i = z_i$, for each i independently, we generate instances of $\mathbb{G}_{*2}(n, \frac{y_i - 3z_i}{2})$ and $\mathbb{G}_{*3}(n, z_i)$. Let $Y'_i = y'_i$ be the number of non-isolated vertices in the constructed instance of $\mathbb{G}_{*2}(n, \frac{y_i - 3z_i}{2}) \cup \mathbb{G}_{*3}(n, z_i)$. By definition $y'_i \leq y_i$. Set now V'_i to be a union of the set of non-isolated vertices of $\mathbb{G}_{*2}(n, \frac{y_i - 3z_i}{2}) \cup \mathbb{G}_{*3}(n, z_i)$ and $y_i - y'_i$ vertices chosen uniformly at random from the remaining ones. This coupling implies

$$\mathbb{G}_{*2}\left(n, \frac{Y_i - 3Z_i}{2}\right) \cup \mathbb{G}_{*3}(n, Z_i) \preceq \mathcal{G}[V_i].$$

Graphs $\mathbb{G}_{*2}(n, \frac{Y_i - 3Z_i}{2}) \cup \mathbb{G}_{*3}(n, Z_i)$, $1 \leq i \leq m$, are independent, and $\mathcal{G}[V_i]$, $1 \leq i \leq m$, are independent. Therefore by Fact 2 and the definition of $\mathcal{G}(n, m, \bar{p})$, we have

$$\begin{aligned}
(11) \quad \mathbb{G}_{*2}(n, M_2) \cup \mathbb{G}_{*3}(n, M_3) &= \bigcup_{1 \leq i \leq m} (\mathbb{G}_{*2}(n, \frac{Y_i - 3Z_i}{2}) \cup \mathbb{G}_{*3}(n, Z_i)) \\
&\preceq \bigcup_{1 \leq i \leq m} \mathcal{G}[V_i] \\
&= \mathcal{G}(n, m, \bar{p}).
\end{aligned}$$

By definition

$$\mathbb{E} \sum_{i=1}^m Y_i = \mathbb{E} \sum_{i=1}^m (X_i - \mathbb{I}_{X_i=1}) = S_1 \quad \text{and} \quad \mathbb{E} \sum_{i=1}^m Z_i = \mathbb{E} \sum_{i=1}^m (\mathbb{I}_{X_i \text{ is odd}} - \mathbb{I}_{X_i=1}) = S_3.$$

Moreover

$$\begin{aligned}
\text{Var} \sum_{i=1}^m Y_i &= \sum_{i=1}^m (\text{Var} X_i + \text{Var} \mathbb{I}_{X_i=1} - 2(\mathbb{E} X_i \mathbb{I}_{X_i=1} - \mathbb{E} X_i \mathbb{E} \mathbb{I}_{X_i=1})) \\
&\leq \sum_{i=1}^m (\mathbb{E} X_i - \mathbb{E} \mathbb{I}_{X_i=1} + 2\mathbb{E} X_i \mathbb{E} \mathbb{I}_{X_i=1}) \\
&= \sum_{i=1}^m (\mathbb{E} Y_i + 2(np_i)^2(1 - p_i)^{n-1}) \\
&\leq \sum_{i=1}^m (\mathbb{E} Y_i + 3(np_i - np_i(1 - p_i)^{n-1})) \\
&\leq 4S_1
\end{aligned}$$

and

$$\begin{aligned}
\text{Var} \sum_{i=1}^m Z_i &= \sum_{i=1}^m (\text{Var} \mathbb{I}_{X_i \text{ odd}} + \text{Var} \mathbb{I}_{X_i=1} - 2(\mathbb{E} \mathbb{I}_{X_i \text{ odd}} \mathbb{I}_{X_i=1} - \mathbb{E} \mathbb{I}_{X_i \text{ odd}} \mathbb{E} \mathbb{I}_{X_i=1})) \\
&\leq \sum_{i=1}^m (\mathbb{E} \mathbb{I}_{X_i \text{ odd}} - \mathbb{E} \mathbb{I}_{X_i=1} + 2\mathbb{E} \mathbb{I}_{X_i \text{ odd}} \mathbb{E} \mathbb{I}_{X_i=1}) \\
&\leq \sum_{i=1}^m (\mathbb{E} X_i - \mathbb{E} \mathbb{I}_{X_i=1} + 2\mathbb{E} X_i \mathbb{E} \mathbb{I}_{X_i=1}) \\
&\leq 4S_1.
\end{aligned}$$

Therefore by Chebyshev's inequality, for any ω' , $1 \ll (\omega')^2 \ll S_3/\sqrt{S_1}$, with high probability

$$\begin{aligned}
(12) \quad \Pr \left\{ \left| \sum_{i=1}^m Y_i - S_1 \right| \geq \omega' \sqrt{S_1} \right\} &\leq \frac{1}{\omega'} = o(1), \\
\Pr \left\{ \left| \sum_{i=1}^m Z_i - S_3 \right| \geq \omega' \sqrt{S_1} \right\} &\leq \frac{S_3}{S_1 \omega'} = o(1).
\end{aligned}$$

Thus with probability $1 - o(1)$

$$\begin{aligned} M_2 &\geq \frac{S_1 - 3S_3 - 4\omega'\sqrt{S_1}}{2}, \\ M_3 &\geq S_3 - \omega'\sqrt{S_1}. \end{aligned}$$

Therefore by Fact 1

$$\begin{aligned} \mathbb{G}_{*2} \left(n, \frac{S_1 - 3S_3 - 4\omega'\sqrt{S_1}}{2} \right) \cup \mathbb{G}_{*3} \left(n, S_3 - \omega'\sqrt{S_1} \right) \\ \preceq_{1-o(1)} \mathbb{G}_{*2} (n, M_2) \cup \mathbb{G}_{*3} (n, M_3) \preceq \mathcal{G} (n, m, \bar{p}). \end{aligned}$$

We may assume that in the above coupling $\mathbb{G}_{*2} \left(n, \frac{S_1 - 3S_3 - 4\omega'\sqrt{S_1}}{2} \right)$ and $\mathbb{G}_{*3} (n, S_3 - \omega'\sqrt{S_1})$ are independent. The main reason for this is the fact that even though M_2 and M_3 are dependent (i.e. also $\mathbb{G}_{*2} (n, M_2)$ and $\mathbb{G}_{*3} (n, M_3)$ are dependent), the choice of a hyperedge of $\mathbb{H}_{*i}(n, \cdot)$ in a given draw in the construction of $\mathbb{G}_{*2} (n, M_2)$ and $\mathbb{G}_{*3} (n, M_3)$ is independent from choices in other draws. Moreover note that in the coupling, in order to get $\mathbb{G}_{*2} (n, M_2) \cup \mathbb{G}_{*3} (n, M_3)$ from a sum of independent graphs $\mathbb{G}_{*2} \left(n, \frac{S_1 - 3S_3 - 4\omega'\sqrt{S_1}}{2} \right) \cup \mathbb{G}_{*3} (n, S_3 - \omega'\sqrt{S_1})$ we may proceed in the following way. Given $M_2 = m_2$ and $M_3 = m_3$:

- if m_2 (or m_3 resp.) is larger than $(S_1 - 3S_3 - 4\omega'\sqrt{S_1})/2$ (or $S_3 - \omega'\sqrt{S_1}$ resp.) then we make $m_2 - (S_1 - 3S_3 - 4\omega'\sqrt{S_1})/2$ (or $m_3 - (S_3 - \omega'\sqrt{S_1})$) additional draws and add hyperedges to $\mathbb{H}_{*2}(n, \frac{S_1 - 3S_3 - 4\omega'\sqrt{S_1}}{2})$ ($\mathbb{H}_{*3}(n, S_3 - \omega'\sqrt{S_1})$ resp.)
- if m_2 (m_3 resp.) is smaller than $(S_1 - 3S_3 - 4\omega'\sqrt{S_1})/2$ (or $S_3 - \omega'\sqrt{S_1}$ resp.) then we delete from $\mathbb{H}_{*2}(n, \frac{S_1 - 3S_3 - 4\omega'\sqrt{S_1}}{2})$ ($\mathbb{H}_{*3}(n, S_3 - \omega'\sqrt{S_1})$ resp.) hypredges attributed to the last draws to get exactly m_i , $i = 2, 3$, draws.

Let M'_2 and M'_3 be random variables with the Poisson distribution

$$\text{Po} \left(\frac{S_1 - 3S_3 - 5\omega'\sqrt{S_1}}{2} \right) \quad \text{and} \quad \text{Po} \left(S_3 - 2\omega'\sqrt{S_1} \right), \text{ respectively.}$$

Then by sharp concentration of the Poisson distribution

$$\Pr \left\{ M'_2 \leq \frac{S_1 - 3S_3 - 4\omega'\sqrt{S_1}}{2} \right\} = 1 - o(1) \quad \text{and} \quad \Pr \left\{ M'_3 \leq S_3 - \omega'\sqrt{S_1} \right\} = 1 - o(1).$$

Therefore by Fact 1, Fact 3 and (7)

$$\begin{aligned} G_2 \left(n, 1 - \exp \left(-\frac{S_1 - 3S_3 - 5\omega'\sqrt{S_1}}{2 \binom{n}{2}} \right) \right) \cup G_3 \left(n, 1 - \exp \left(-\frac{S_3 - 2\omega'\sqrt{S_1}}{\binom{n}{3}} \right) \right) \\ = \mathbb{G}_{*2} (n, M'_2) \cup \mathbb{G}_{*3} (n, M'_3) \\ \preceq_{1-o(1)} \mathbb{G}_{*2} \left(n, \frac{S_1 - 3S_3 - 4\omega'\sqrt{S_1}}{2} \right) \cup \mathbb{G}_{*3} (n, S_3 - \omega'\sqrt{S_1}) \\ \preceq_{1-o(1)} \mathcal{G} (n, m, \bar{p}). \end{aligned}$$

For $S_1 = o(n^2)$ (then also $S_3 \leq S_1 = o(n^3)$), $\omega = 5\omega'$, and \hat{p}_2 and \hat{p}_3 as in (2)

$$\hat{p}_2 = \frac{S_1 - 3S_3 - 5\omega'\sqrt{S_1} - \frac{2S_3^2}{n^2}}{2\binom{n}{2}} \leq 1 - \exp\left(-\frac{S_1 - 3S_3 - 5\omega'\sqrt{S_1}}{2\binom{n}{2}}\right);$$

$$\hat{p}_3 \leq \frac{S_3 - 2\omega'\sqrt{S_1} - \frac{2S_3^2}{n^3}}{\binom{n}{3}} \leq 1 - \exp\left(-\frac{S_3 - 2\omega'\sqrt{S_1}}{\binom{n}{3}}\right).$$

Therefore using standard couplings of $G_2(n, \cdot)$ and $H_3(n, \cdot)$ finally we get

$$\begin{aligned} & G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3) \\ & \preceq G_2\left(n, 1 - \exp\left(-\frac{S_1 - 3S_3 - 5\omega'\sqrt{S_1}}{2\binom{n}{2}}\right)\right) \cup G_3\left(n, 1 - \exp\left(-\frac{S_3 - 2\omega'\sqrt{S_1}}{\binom{n}{3}}\right)\right) \\ & = \mathbb{G}_{*2}\left(n, \text{Po}\left(\frac{S_1 - 3S_3 - 5\omega'\sqrt{S_1}}{2}\right)\right) \cup \mathbb{G}_{*3}\left(n, \text{Po}\left(S_3 - 2\omega'\sqrt{S_1}\right)\right) \\ & \preceq_{1-o(1)} \mathcal{G}(n, m, \bar{p}). \end{aligned}$$

Therefore the result follows by Fact 4. □

4 Vertex degrees in $\mathcal{G}(n, m, \bar{p})$

For any graph G denote by $\delta(G)$ the minimum vertex degree in G .

Lemma 1. *Let $\max_{1 \leq i \leq m} p_i = o((\ln n)^{-1})$ and S_1 be given by (1). If*

$$S_1 = n(\ln n + c_n)$$

then

$$\lim_{n \rightarrow \infty} \Pr\{\delta(\mathcal{G}(n, m, \bar{p})) \geq 1\} = f(c_n),$$

where $f(\cdot)$ is given by (5).

Note that the condition $\max_{1 \leq i \leq m} p_i = o((\ln n)^{-1})$ is necessary. Otherwise the number of vertices of a given degree depends more on the fluctuations of the values of the vector \bar{p} . For example let $m = n^2$ and \bar{p} equals

$$\left(\underbrace{\frac{b_n}{\ln n}, \dots, \frac{b_n}{\ln n}}_{\ln n(\ln n + c_n)/b_n}, \frac{1}{nm}, \dots, \frac{1}{nm}\right) \quad \text{or} \quad \left(\frac{1}{\sqrt{\ln n}}, \sqrt{\frac{\ln n + c_n}{nm}}, \dots, \sqrt{\frac{\ln n + c_n}{nm}}\right),$$

for some $b_n, c_n = o(\ln n)$. In both cases

$$\sum_{i=1}^m np_i (1 - (1 - p_i)^{n-1}) = n(\ln n + c_n + o(1))$$

but the expected number of vertices of degree 0 in $\mathcal{G}(n, m, \bar{p})$ is

$$(1 + o(1)) \exp\left(-c_n - \frac{b_n}{2}\right) \quad \text{and} \quad (1 + o(1)) \exp(-c_n),$$

respectively.

Lemma 2. *Let $\max_{1 \leq i \leq m} p_i = o(\ln n^{-1})$, k be a positive integer, S_1 and $S_{1,t}$, $t = 2, \dots, k$ be given by (1) and, $c_n \rightarrow \infty$.*

(i) *If*

$$S_1 = n \left(\ln n + (k-1) \ln \left(\max \left\{ 1, \left(\frac{S_{1,2}}{S_1} \ln n \right) \right\} \right) - c_n \right)$$

then

$$\lim_{n \rightarrow \infty} \Pr \{ \delta(\mathcal{G}(n, m, \bar{p})) \geq k \} = 0$$

(ii) *If*

$$S_1 - \sum_{t=3}^k S_{1,t} = n \left(\ln n + (k-1) \ln \left(\max \left\{ 1, \left(\frac{S_{1,2}}{S_1} \ln n \right) \right\} \right) + c_n \right)$$

then

$$\lim_{n \rightarrow \infty} \Pr \{ \delta(\mathcal{G}(n, m, \bar{p})) \geq k \} = 1.$$

Here and in the proof we assume that $\sum_{t=3}^2 S_{1,t} = 0$. For $\mathcal{G}(n, m, p)$ the result of Lemma 2 may be improved.

Lemma 3. *Let k be a positive integer and $m \gg \ln^2 n$. If*

$$p(1 - (1-p)^{n-1}) = \frac{\ln n + \ln \left(\max \left\{ 1, (np)^{k-1} \left(\left(\frac{e^{-np} \ln n}{1 - e^{-np}} \right)^{k-1} + \frac{e^{-np} \ln n}{1 - e^{-np}} \right) \right\} \right) + c_n}{m},$$

then

$$\lim_{n \rightarrow \infty} \Pr \{ \delta(\mathcal{G}(n, m, p)) \geq k \} = \begin{cases} 0 & \text{for } c_n \rightarrow -\infty; \\ 1 & \text{for } c_n \rightarrow \infty. \end{cases}$$

Proof of Lemma 1. In the proof we assume that $c_n = o(\ln n)$. Note that if $\bar{p}' = (p'_1, \dots, p'_m)$ and $\bar{p} = (p_1, \dots, p_m)$ are such that $p'_i \leq p_i$, for all $1 \leq i \leq m$, then $\mathcal{G}(n, m, \bar{p}') \preceq \mathcal{G}(n, m, \bar{p})$ and $\sum_{i=1}^m np'_i(1 - (1 - p'_i)^{n-1}) \leq \sum_{i=1}^m np_i(1 - (1 - p_i)^{n-1})$. Therefore by monotonicity of the considered property, analysis of the case $c_n = o(\ln n)$ is enough to prove the general result stated above.

Consider a coupon collector process in which in each draw one choose one coupon uniformly at random from \mathcal{V} . In order to determine the minimum degree we establish a coupling of the coupon collector process on \mathcal{V} and the construction of $\mathcal{G}(n, m, \bar{p})$. Define V'_i and Y_i as in (9) and (10). We consider the process in which we collect coupons from \mathcal{V} and at the same time we construct a family of sets $\{V'_i : i = 1, \dots, m\}$ (i.e. equivalently we construct an instance of $\mathcal{G}(n, m, \bar{p})$). Assume we have a given vector

$$(Y_1, Y_2, \dots, Y_m) = (y_1, y_2, \dots, y_m)$$

chosen so that Y_i are independent random variables with distribution of the random variables defined in (10). We divide the process of collecting coupons into m phases. In the i -th phase, $1 \leq i \leq m$, we draw independently one by one vertices uniformly at random from \mathcal{V} until within the phase we get y_i distinct vertices. Let V'_i be the set of vertices chosen in the i -th phase. We construct an instance of $\mathcal{G}(n, m, \bar{p})$ by connecting by edges all pairs of vertices within V'_i , for all $1 \leq i \leq m$. Obviously after the m -th phase $\bigcup_{1 \leq i \leq m} V'_i$ is the set of non-isolated vertices in $\mathcal{G}(n, m, \bar{p})$. Denote by T_i the number of vertices drawn in the i -th phase. If $Y_i = y_i = 0$ then $T_i = 0$. If $Y_i = y_i \geq 2$ and in the i -th phase we have already collected $j < y_i$ vertices then the number of draws to collect the $j+1$ -st vertex has geometric distribution with parameter $\frac{n-j}{n}$. Therefore

$$\mathbb{E}(T_i - Y_i | Y_i = y_i) = \sum_{j=0}^{y_i-1} \frac{n}{n-j} - y_i = \sum_{j=0}^{y_i-1} \frac{j}{n-j} \leq \begin{cases} \sum_{j=0}^{y_i-1} \frac{2j}{n} = \frac{y_i(y_i-1)}{n} & \text{for } 2 \leq y_i < n/2; \\ n \ln n & \text{for } y_i \geq n/2. \end{cases}$$

Note that

$$\Pr \left\{ Y_i \geq \frac{n}{2} \right\} \leq \binom{n}{\frac{n}{2}} p_i^{\frac{n}{2}} \leq \left(\frac{enp_i}{\frac{n}{2}} \right)^{\frac{n}{2}} \leq \frac{p_i^2}{\ln n}.$$

Thus

$$\mathbb{E}(T_i - Y_i) \leq \sum_{y_i=2}^{n/2} \frac{y_i(y_i-1)}{n} \binom{n}{y_i} p_i^{y_i} (1-p_i)^{n-y_i} + n \ln n \Pr \left\{ Y_i \geq \frac{n}{2} \right\} \leq 2np_i^2.$$

Therefore

$$\begin{aligned} \mathbb{E} \left(\sum_{i=1}^m (T_i - Y_i) \right) &\leq \sum_{i=1}^m 2np_i^2 \\ &= \sum_{i=1}^m 2 \max \left\{ p_i, \frac{1}{n} \right\} \min \{ np_i, n^2 p_i^2 \} \\ &\leq \frac{4}{\min_{1 \leq i \leq m} \{ p_i^{-1}, n \}} \sum_{i=1}^m \frac{1}{2} \min \{ np_i, n^2 p_i^2 \} \\ &= o \left(\frac{1}{\ln n} \right) \sum_{i=1}^m np_i (1 - (1-p_i)^{n-1}) \\ &= o \left(\frac{S_1}{\ln n} \right), \end{aligned}$$

where S_1 is defined as in (1). Moreover by Markov's inequality for any ω

$$\Pr \left\{ \sum_{i=1}^m (T_i - Y_i) \geq \frac{S_1}{\omega \ln n} \right\} \leq \mathbb{E} \left(\sum_{i=1}^m (T_i - Y_i) \right) \frac{\omega \ln n}{S_1}.$$

Therefore with high probability

$$(13) \quad \sum_{i=1}^m (T_i - Y_i) \leq \frac{S_1}{\omega \ln n}, \text{ for any } 1 \ll \omega \ll \frac{S_1}{(\ln n \mathbb{E}(\sum_{i=1}^m (T_i - Y_i)))}.$$

Given $1 \ll \omega \ll \min\{S_1/(\ln n \mathbb{E}(\sum_{i=1}^m (T_i - Y_i))); \sqrt{S_1}/\ln n\}$ let

$$T_- = S_1 - \omega\sqrt{S_1} \quad \text{and} \quad T_+ = S_1 + \omega\sqrt{S_1} + \frac{S_1}{\omega \ln n}.$$

In the probability space of the coupling described above define events:

- \mathcal{A}_- – all coupons are collected in at most T_- draws;
- \mathcal{A}_+ – all coupons are collected in at most T_+ draws;
- \mathcal{A} – $\delta(\mathcal{G}(n, m, \bar{p})) \geq 1$;
- \mathcal{B} – the construction of $\mathcal{G}(n, m, \bar{p})$ is finished between T_- -th and T_+ -th draw;
- \mathcal{B}_1 – the construction of $\mathcal{G}(n, m, \bar{p})$ is finished in at most $\sum_{i=1}^m Y_i + S_1/(\omega \ln n)$;
- \mathcal{B}_2 – in $\mathcal{G}(n, m, \bar{p})$ we have $S_1 - \omega\sqrt{S_1} \leq \sum_{i=1}^m Y_i \leq S_1 + \omega\sqrt{S_1}$.

By definition

$$\mathcal{A}_+ \cap \mathcal{B} \subseteq \mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}_- \cap \mathcal{B}.$$

Therefore

$$(14) \quad \Pr\{\mathcal{A}_+\} - \Pr\{\mathcal{B}_1^c\} \leq \Pr\{\mathcal{A}\} \leq \Pr\{\mathcal{A}_-\} + \Pr\{\mathcal{B}^c\}.$$

Let $S_1 = n(\ln n + c_n)$, then

$$T_{\pm} = n(\ln n + c_n + o(1)).$$

Therefore by the classical results on the coupon collector problem

$$\Pr\{\mathcal{A}_-\} \sim \Pr\{\mathcal{A}_+\} \sim f(c_n)$$

and by (12) and (13)

$$\Pr\{\mathcal{B}^C\} = \Pr\{\mathcal{B}_1^C \cup \mathcal{B}_2^C\} = 1 - o(1).$$

Thus the lemma follows by substituting the above values to (14). \square

The proofs of Lemmas 2 and 3 rely on a technique of dividing $\mathcal{G}(n, m, \bar{p})$ into subgraphs. We gather here some simple facts concerning this division. For each $2 \leq t \leq k$, let $\mathcal{G}_t(n, m, \bar{p})$ be a random graph with a vertex set \mathcal{V} and an edge set consisting of those edges from $\mathcal{G}(n, m, \bar{p})$, which are contained in at least one of the sets $\{V_i : |V_i| = t\}$. Moreover let $\mathcal{G}_{k+1}(n, m, \bar{p})$ be a subgraph of $\mathcal{G}(n, m, \bar{p})$ containing only those edges which are subsets of at least one of the sets $\{V_i : |V_i| \geq k+1\}$. Define

$$M_t = \sum_{i=1}^m \mathbb{I}_{Y_i=t}, \quad \text{for all } t = 2, \dots, k, \quad \text{and} \quad M_{k+1} = \sum_{i=1}^m Y_i - \sum_{t=2}^k t M_t.$$

Note that, for all $t = 2, \dots, k$, we have $\mathcal{G}_t(n, m, \bar{p}) = \mathbb{G}_{*t}(n, M_t)$, where $\mathbb{G}_{*t}(n, \cdot)$ is defined as in Section 3. Moreover

$$\mathbb{E}M_t = \frac{S_{1,t}}{t} \quad \text{for all } t = 2, \dots, k-1, \quad \text{and} \quad \mathbb{E}M_{k+1} = S_1 - \sum_{t=2}^k S_{1,t},$$

where S_1 and $S_{1,t}$ are defined as in (1). For all $1 \leq t \leq k$, M_t is a sum of independent Bernoulli random variables and $S_{1,t} \leq S_1$. Therefore by Chebyshev's inequality, Markov's inequality, and by (12) for any $\omega \rightarrow \infty$ with high probability

$$(15) \quad \max \left\{ 0, \frac{S_{1,t} - \omega \sqrt{S_1}}{t} \right\} \leq M_t \leq \frac{S_{1,t} + \omega \sqrt{S_1}}{t}, \quad \text{for all } t = 2, 3, \dots, k$$

$$(16) \quad M_{k+1} \geq S_1 - \sum_{t=2}^k S_{1,t} - k\omega \sqrt{S_1},$$

$$(17) \quad M_{k+1} + \sum_{j=2}^k jM_j - tM_t \leq S_1 - S_{1,t} + 2\omega \sqrt{S_1}, \quad \text{for all } t = 2, 3, \dots, k.$$

Moreover, for any $t = 2, 3, \dots, k$, if $S_{1,t} \rightarrow \infty$ then with high probability

$$(18) \quad \frac{S_{1,t} - \omega \sqrt{S_{1,t}}}{t} \leq M_t \leq \frac{S_{1,t} + \omega \sqrt{S_{1,t}}}{t}.$$

Set ω a function tending slowly to infinity. In the proofs we will assume that ω is small enough to get the needed bounds.

For $t = 2, \dots, k$ let

$$(19) \quad \begin{aligned} T_{t+} &= S_1 - S_{1,t} + 2\omega \sqrt{S_1} + \frac{S_1}{\omega \ln n}; \\ T_- &= \max \left\{ 0, S_1 - \sum_{t=2}^k S_{1,t} - k\omega \sqrt{S_1}, \right\}; \\ \hat{p}_{t-} &= \frac{S_{1,t} - 2\omega \sqrt{S_{1,t}} - t! S_{1,t}^2 n^{-t}}{t \binom{n}{t}}; \\ \hat{p}_{t+} &= \frac{S_{1,t} + 2\omega \sqrt{S_{1,t}}}{t \binom{n}{t}}; \\ \hat{q}_t &= \begin{cases} 0, & \text{for } S_{1,t} = O(\omega \sqrt{S_1}) \\ \frac{S_{1,t} - 2\omega \sqrt{S_{1,t}} - t! S_{1,t}^2 n^{-2}}{t \binom{n}{t}}, & \text{otherwise.} \end{cases} \end{aligned}$$

For all $t = 2, \dots, k$ $\mathcal{G}_t(n, m, \bar{p}) = \mathbb{G}_{*t}(n, M_2)$, thus by (15) and (18), using the same methods as in the proof of Theorem 1 we can show the following fact.

Fact 5. *Let \hat{q}_t and $\hat{p}_{t\pm}$ be defined as in (19). Then*

$$G_t(n, \hat{q}_t) \preceq_{1-o(1)} \mathcal{G}_t(n, m, \bar{p}),$$

where $G_t(n, \hat{q}_t)$ is independent from $(\mathcal{G}_j(n, m, \bar{p}))_{j=2, \dots, k+1, j \neq t}$.

Moreover if $S_{1,t} \rightarrow \infty$ then

$$G_t(n, \hat{p}_{t-}) \preceq_{1-o(1)} \mathcal{G}_t(n, m, \bar{p}) \preceq_{1-o(1)} G_t(n, \hat{p}_{t+}).$$

where $G_t(n, \hat{p}_{t\pm})$ is independent from $(\mathcal{G}_j(n, m, \bar{p}))_{j=2, \dots, k+1, j \neq t}$.

As in the proof of Lemma 1 we will use the coupling of the coupon collector process on \mathcal{V} and the construction of $\mathcal{G}(n, m, \bar{p})$. However here, for some $t = 2, \dots, k$, in the process we omit rounds in which $Y_i = t$. Therefore we construct $\bigcup_{j=2, \dots, k+1; j \neq t} \mathcal{G}_j(n, m, \bar{p})$ instead of $\mathcal{G}(n, m, \bar{p})$. Reasoning in the same way as in the proof of Lemma 1, by the definition of M_t , (13), and (17) we get that with high probability the construction of $\bigcup_{j=2, \dots, k+1; j \neq t} \mathcal{G}_j(n, m, \bar{p})$ is finished before the T_{t+} -th draw. Similarly, if in the process we omit rounds with $Y_i \leq k$, then with high probability we construct $\mathcal{G}_{k+1}(n, m, \bar{p})$ in at least T_- draws. Therefore analogous reasoning as this used in the proof of Lemma 1 gives the following facts.

Fact 6. *There exists a coupling of the coupon collector process with the set of coupons \mathcal{V} and the construction of $\bigcup_{j=2, \dots, k+1; j \neq t} \mathcal{G}_j(n, m, \bar{p})$ such that with high probability the number of collected coupons in T_{t+} draws is at least the number of non-isolated vertices in $\bigcup_{j=2, \dots, k+1; j \neq t} \mathcal{G}_j(n, m, \bar{p})$.*

Fact 7. *There exists a coupling of the coupon collector process with the set of coupons \mathcal{V} and the construction of $\mathcal{G}_{k+1}(n, m, \bar{p})$ such that with high probability the number of collected coupons in T_- draws is at most the number of non-isolated vertices in $\mathcal{G}_{k+1}(n, m, \bar{p})$.*

Proof of Lemma 2. In the proof we use notation introduced in (19). Moreover let $a_n = S_{1,2}/S_1$.

(i) We restrict our attention to the case $c_n = o(\ln n)$. In the latter cases the result follows by Lemma 1.

First consider the case $a_n = S_{1,2}/S_1 \gg 1/\ln n$. Then $S_{1,2} \rightarrow \infty$. Take any probability space on which we define the coupon collector process on \mathcal{V} and $G_2(n, \hat{p}_{2+})$, in such a manner that they are independent. Let X_+ be a random variable counting vertices which have not been collected during the coupon collector process in T_{2+} draws and have degree at most $k-1$ in $G_2(n, \hat{p}_{2+})$. If

$$S_1 = n(\ln n + (k-1) \ln(a_n \ln n) - c_n)$$

then for ω tending to infinity slowly enough

$$\frac{1}{n} T_{2+} + (n-1) \hat{p}_{2+} = \frac{1}{n} \left(S_1 + O \left(\omega \sqrt{S_1} + \frac{S_1}{\omega \ln n} \right) \right) = \ln n + (k-1) \ln(a_n \ln n) - c_n + o(1)$$

and

$$\hat{p}_{2+} \sim \frac{a_n \ln n}{n}.$$

Therefore

$$\begin{aligned} \mathbb{E}X_+ &= n \left(1 - \frac{1}{n} \right)^{T_{2+}} \left(\sum_{i=0}^{k-1} \binom{n-1}{i} \hat{p}_{2+}^i (1 - \hat{p}_{2+})^{k-1-i} \right) (1 - \hat{p}_{2+})^{n-k} \\ &\sim n \frac{1}{(k-1)!} (a_n \ln n)^{k-1} \exp(-\ln n - (k-1) \ln(a_n \ln n) - c_n) \sim \frac{e^{-c_n}}{(k-1)!} \end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}X_+(X_+ - 1) &\sim n^2 \left(1 - \frac{2}{n}\right)^{T_{2+}} \left(\sum_{i=0}^{k-1} \binom{n-2}{i} \hat{p}_{2+}^i (1 - \hat{p}_{2+})^{k-1-i}\right)^2 (1 - \hat{p}_{2+})^{2n-2k-1} \\
&\quad + n^2 \left(1 - \frac{2}{n}\right)^{T_{2+}} \hat{p}_{2+} \left(\sum_{i=0}^{k-2} \binom{n-2}{i} \hat{p}_{2+}^i (1 - \hat{p}_{2+})^{k-2-i}\right)^2 (1 - \hat{p}_{2+})^{2n-2k} \\
&\sim \left(\frac{e^{-c_n}}{(k-1)!}\right)^2.
\end{aligned}$$

Thus by the second moment method with high probability $X_+ > 0$ as $c_n \rightarrow -\infty$. Note that if there is a vertex which is isolated in $\bigcup_{t=3}^k \mathcal{G}_t(n, m, \bar{p})$ and has degree at most $k-1$ in $\mathcal{G}_2(n, m, \bar{p})$, then $\delta(\mathcal{G}(n, m, \bar{p})) \leq k-1$. By Facts 6 and 5 there is a probability space such that with high probability $\mathcal{G}_2(n, m, \bar{p}) \subseteq G_2(n, \hat{p}_{2+})$, the number of isolated vertices in $\bigcup_{t=3}^k \mathcal{G}_t(n, m, \bar{p})$ is at least the number of non-collected coupons after T_{2+} draws, and $G_2(n, \hat{p}_{2+})$ and the coupon collector process are independent. Thus $X_+ > 0$ imply that with high probability $\delta(\mathcal{G}(n, m, \bar{p})) \leq k-1$.

Now let $S_{1,2}/S_1 = O((\ln n)^{-1})$. Then $S_1 = n(\ln n + c_n + O(1))$. Therefore for the result follows by Lemma 1.

(ii) In the proof we restrict our attention to the case $c_n = O(\ln n)$. In the latter case the result follows after combining (3) with known results on $G_2(n, \hat{p})$.

Let $a_n = S_{1,2}/S_1 \gg 1/\ln n$ (thus $S_{1,2} \rightarrow \infty$ and $a_n \rightarrow \infty$). Consider any probability space on which we define the coupon collector process on \mathcal{V} and $G_2(n, \hat{p}_{2-})$, in such a manner that they are independent. Let X_- be a random variable defined on this probability space and counting vertices which have not been collected during the coupon collector process with T_- draws and have degree at most $k-1$ in $G_2(n, \hat{p}_{2-})$. If $S_1 - \sum_{t=3}^k S_{1,t} = n(\ln n + (k-1)\ln(a_n \ln n) + c_n)$ then for ω tending to infinity slowly enough

$$\begin{aligned}
\frac{1}{n}T_- + (n-1)\hat{p}_{2-} &\geq \frac{1}{n} \left(S_1 - \sum_{t=3}^{k-1} S_{1,t} + O\left(\omega\sqrt{S_1} + S_1^2 n^{-2}\right) \right) \\
&= \ln n + (k-1)\ln(a_n \ln n) + c_n + o(1).
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}X_- &= n \left(1 - \frac{1}{n}\right)^{T_-} \left(\sum_{i=0}^{k-1} \binom{n-1}{i} \hat{p}_{2-}^i (1 - \hat{p}_{2-})^{k-1-i}\right) (1 - \hat{p}_{2-})^{n-k} \\
&= O(1)n \frac{1}{(k-1)!} (a_n \ln n)^{k-1} \exp(-\ln n - (k-1)\ln(a_n \ln n) - c_n) = O(1) \frac{e^{-c_n}}{(k-1)!}
\end{aligned}$$

Therefore with high probability $X_- = 0$, i.e. with high probability each vertex is collected in T_- draws or has degree at least k in $G_2(n, \hat{p}_{2-})$. Note that if each vertex is non-isolated in $\mathcal{G}_{k+1}(n, m, \bar{p})$ or has degree at least k in $\mathcal{G}_2(n, m, \bar{p})$, then $\delta(\mathcal{G}(n, m, \bar{p})) \geq k$. Therefore Facts 5 and 7 imply that with high probability $\delta(\mathcal{G}(n, m, \bar{p})) \geq k$.

Now let $S_{1,2}/S_1 = O((\ln n)^{-1})$, i.e. $S_{1,2} = O(n)$. Then $S_1 - \sum_{t=2}^{k-1} S_{1,t} = n(\ln n + c_n + O(1))$ and

$$T_- = n(\ln n + c_n + O(1)).$$

Thus by Fact 7 with high probability there is no isolated vertex in $\mathcal{G}_{k+1}(n, m, \bar{p})$, i.e. with high probability $\delta(\mathcal{G}(n, m, \bar{p})) \geq k$. \square

Proof of Lemma 3. We use notation from (19). Moreover let

$$a_n = (np)^{k-1} \left(\left(\frac{e^{-np} \ln n}{1 - e^{-np}} \right)^{k-1} + \frac{e^{-np} \ln n}{1 - e^{-np}} \right).$$

Note that

$$S_{1,t} \sim m \frac{(np)^t}{(t-1)!} (1-p)^n \sim ((t-1)!)^{-1} \frac{(np)^{t-1} (1-p)^n}{1 - (1-p)^{n-1}} S_1, \quad \text{for } t = 2, \dots, k,$$

where S_1 and $S_{1,t}$ are defined as in (1). Therefore as far as $np^2 = o(1)$

$$\left(\frac{S_{1,2}}{S_1} \ln n \right)^{k-1} + \frac{S_{1,k}}{S_1} \ln n \sim a_n.$$

If $S_{1,t} = O(S_1/\ln n)$ for all t then $a_n = O(1)$ and $\frac{S_{1,2}}{S_1} \ln n = O(1)$. Therefore the lemma follows by Lemma 2. It remains to consider the case: $S_{1,2} \gg S_1/\ln n$ or $S_{1,k} \gg S_1/\ln n$. Note that in this case if $c_n = O(\ln n)$ then $np^2 = o(1)$ and $a_n \rightarrow \infty$.

Let now $c_n \rightarrow -\infty$. As in the proof of Lemma 2 we may restrict our attention to $c_n = o(\ln n)$. If $e^{-np} \ln n / (1 - e^{-np}) = \Omega(1)$, then the lemma follows by Lemma 2, therefore assume that

$$(20) \quad \frac{e^{-np} \ln n}{1 - e^{-np}} = o(1) \quad (\text{i.e. } a_n \sim (np)^{k-1} e^{-np} \ln n).$$

By (20) $S_{1,k} \rightarrow \infty$, thus we may apply the second part of Fact 5.

Take any probability space on which we define independent coupon collector process on \mathcal{V} and $G_k(n, \hat{p}_{k+})$. Let X_+ be a random variable counting vertices which have not been collected during the coupon collector process with T_{k+} draws and have degree at most $k-1$ in $G_k(n, \hat{p}_{k+})$. For ω tending to infinity slowly enough

$$\frac{1}{n} T_{k+} + \binom{n-1}{k-1} \hat{p}_{k+} = \frac{1}{n} \left(S_1 + O \left(\omega \sqrt{S_1} + S_1^2 n^{-2} + \frac{S_1}{\omega \ln n} \right) \right) = \ln n + \ln a_n + c_n + o(1),$$

thus

$$\begin{aligned} \mathbb{E} X_+ &= n \left(1 - \frac{1}{n} \right)^{T_{k+}} \left((1 - \hat{p}_{k+})^{\binom{n-1}{k-1}} + \binom{n-1}{k-1} \hat{p}_{k+} (1 - \hat{p}_{k+})^{\binom{n-1}{k-1}-1} \right) \\ &\sim n \frac{1}{(k-1)!} a_n \exp(-\ln n - \ln a_n - c_n) \sim \frac{e^{-c_n}}{(k-1)!} \end{aligned}$$

and

$$\begin{aligned}\mathbb{E}X_+(X_+ - 1) &\sim n^2 \left(1 - \frac{2}{n}\right)^{T_{k+}} \left(1 + \binom{n-2}{k-1} \hat{p}_{k+} (1 - \hat{p}_{k+})^{-1}\right)^2 (1 - \hat{p}_{k+})^{2\binom{n-1}{k-1} - \binom{n-2}{k-2}} \\ &\quad + n^2 \left(1 - \frac{2}{n}\right)^{T_{k+}} \binom{n-2}{k-2} \hat{p}_{k+} (1 - \hat{p}_{k+})^{2\binom{n-1}{k-1} - \binom{n-2}{k-2} - 1} \\ &\sim \left(\frac{e^{-c_n}}{(k-1)!}\right)^2.\end{aligned}$$

Thus with high probability $X_+ > 0$ as $c_n \rightarrow -\infty$ (i.e. with high probability there is a vertex, which has degree at most $k-1$ in $G_k(n, \hat{p}_{k+})$ or has not been collected in T_{k+} draws). Thus by Facts 5 and 6 with high probability $\delta(\mathcal{G}(n, m, p)) \leq k-1$.

Now assume that $c_n \rightarrow \infty$. As it is explained in the proof of Lemma 2 we may restrict our considerations to the case $c_n = O(\ln n)$. Recall that we assume that $S_{1,2} \gg S_1/\ln n$ or $S_{1,t} \gg S_1/\ln n$, i.e. $a_n \rightarrow \infty$. Note that by definition

$$\hat{q}_t \leq \frac{S_{1,t}}{t \binom{n}{t}} = O(mp^t e^{-np}).$$

Take any probability space on which we may define independent coupon collector process and $\bigcup_{t=2}^k G_t(n, \hat{q}_t)$. Let X_- be a random variable counting vertices which have not been collected during the coupon collector process in T_- draws and have degree at most $k-1$ in $\bigcup_{t=2}^k G_t(\hat{q}_t)$. Note that if v has degree at most $k-1$ in $\bigcup_{t=2}^k G_t(\hat{q}_t)$, then for some $0 \leq k_0 \leq k-1$ and a sequence of integers r_2, \dots, r_t such that $\sum_{t=2}^k (t-1)r_t = k_0$

- (i) there is a set $\mathcal{V}' \subseteq \mathcal{V}$ of k_0 vertices such that for each t there are r_t hyperedges in $H_t(\hat{q}_t)$ (generating $G_t(\hat{q}_t)$) contained in $\mathcal{V}' \cup \{v\}$.
- (ii) and all hyperedges in $\bigcup_{t=2}^k H_t(\hat{q}_t)$ containing v are subsets of $\mathcal{V}' \cup \{v\}$.

Let $r = \sum_{t=2}^k r_t$, then for $c_n = O(\ln n)$ event (i) occurs with probability

$$\begin{aligned}&O(1)n^{k_0} \prod_{t=2}^k ((mp^t)e^{-np})^{r_t} \\ &= O(1)n^{k_0} m^r p^{k_0+r} (e^{-np})^r \\ &= O(1)(np)^{k_0} (mpe^{-np})^r \\ &= O(1)(np)^{k_0} \left(\frac{\ln n}{(1-e^{-np})} e^{-np}\right)^r \\ &= O(a_n)\end{aligned}$$

Moreover

$$\begin{aligned}\frac{1}{n}T_- + \sum_{t=2}^k \binom{n-1}{t-1} \hat{q}_t &\geq \frac{1}{n} \left(S_1 - \sum_{t=3}^{k-1} S_{1,t} + O\left(\omega\sqrt{S_1} + S_1^2 n^{-2}\right)\right) \\ &= \ln n + \ln a_n + c_n + o(1).\end{aligned}$$

Thus

$$\mathbb{E}X_- = n \left(1 - \frac{1}{n}\right)^{T_-} O(a_n) \prod_{t=2}^k (1 - \hat{q}_t)^{\binom{n-1}{t-1} - \binom{k-1}{t-1}} = o(1).$$

Therefore with high probability $X_- = 0$. By Facts 5 and 7 with high probability $\delta(\mathcal{G}(n, m, p)) \geq k$. \square

5 Structural properties of $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$

For any graph G and any set $S \subseteq V(G)$ denote by N_G the set of neighbours of vertices from S contained in $V(G) \setminus S$. For simplicity we write $N_2(\cdot) = N_{G_2(n, \hat{p}_2)}(\cdot)$ and $N_3(\cdot) = N_{G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)}(\cdot)$. Note that $|N_2(S)| \leq |N_3(S)|$.

We first show that $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ with high probability have several structural properties.

Lemma 4. *Let k and C be positive integers and*

$$\begin{aligned} \hat{p}_2 + \frac{n}{2}\hat{p}_3 &= \frac{\theta_n \ln n}{n} \quad \text{for } \theta_n \rightarrow \theta > \frac{1}{2}, \\ \hat{p}_2 &= \frac{\theta'_n \ln n}{n} \quad \text{for } \theta'_n \rightarrow \theta' > \frac{1}{2} \end{aligned}$$

and $\gamma > 1 - \theta'$. Then with high probability $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ has the following properties

(i) \mathcal{B}_1 – for all $S \subseteq \mathcal{V}$ such that $n^\gamma \leq |S| \leq \frac{1}{4}n$

$$|N_3(S)| > 2|S|.$$

(ii) \mathcal{B}_2 – for all $S \subseteq \mathcal{V}$ such that $n^\gamma \leq |S| \leq \frac{2}{3}n$

$$|N_3(S)| > \min\{|S|, 4n \ln \ln n / \ln n\} > \min\{|S|, \alpha(G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3))\},$$

where $\alpha(G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3))$ is the stability number of $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$.

(iii) $\mathcal{B}_{3,k}$ – for all $S \subseteq \mathcal{V}$ if $1 \leq |S| \leq n^\gamma$ and all vertices in S have degree at least $4k + 15$ in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ we have

$$|N_3(S)| \geq 2k|S|.$$

(iv) $\mathcal{B}_{4,C}$ – any two vertices of degree at most C are at distance at least 6.

(v) \mathcal{B}_5 – contains a path of length at least

$$\left(1 - \frac{8 \ln 2}{\ln n}\right).$$

Proof. (i) and (ii) By Chernoff's inequality (for the proof see for example Theorem 2.1 in [10]) for any $\delta = o(1)$ and any binomial random variable X we have

$$\Pr \{X \leq \delta \mathbb{E}X\} \leq \exp(-\mathbb{E}X(\delta \ln \delta + 1 - \delta)) = \exp(-(1 + o(1))\mathbb{E}X).$$

First consider the case $n^\gamma \leq |S| \leq 4n \ln \ln n / \ln n$. Let $s = |S|$. Then $|N_2(S)|$ has the binomial distribution $\text{Bin}(n - s, 1 - (1 - \hat{p}_2)^s)$ and $\mathbb{E}|N_2(S)| \gg 2|S|$. Denote by X the number of sets S of cardinality $n^\gamma \leq |S| \leq 4n \ln \ln n / \ln n$ such that $|N_2(S)| \leq 2|S|$. Using Chernoff's inequality.

$$\begin{aligned} \mathbb{E}X &\leq \sum_{s=n^\gamma}^{4n \ln \ln n / \ln n} \binom{n}{s} \exp(-(1 + o(1))(n - s)(1 - (1 - \hat{p}_2)^s)) \\ &\leq \sum_{s=n^\gamma}^{n/(\ln n \ln \ln n)} \exp\left(s\left(1 + \ln \frac{n}{s}\right) - (1 + o(1))sn\hat{p}_2\right) \\ &\quad + \sum_{s=n/(\ln n \ln \ln n)}^{4n \ln \ln n / \ln n} \exp\left(s\left(1 + \ln \frac{n}{s}\right) - n\frac{\theta'_n}{2 \ln \ln n}\right) \\ &\leq \sum_{s=n^\gamma}^{n/(\ln n \ln \ln n)} \exp(s(1 + (1 - \gamma) \ln n - (1 + o(1))\theta'_n \ln n)) \\ &\quad + \sum_{s=n/(\ln n \ln \ln n)}^{4n \ln \ln n / \ln n} \exp\left(s\left((1 + o(1)) \ln \ln n - \frac{\theta'_n \ln n}{8(\ln \ln n)^2}\right)\right) \\ &= o(1). \end{aligned}$$

Therefore with high probability for all sets $S \subseteq \mathcal{V}$ such that $n^\gamma \leq |S| \leq 4n \ln \ln n / \ln n$ we have

$$|N_2(S)| \geq 2|S|.$$

Now consider the case $|S| \geq 4n \ln \ln n / \ln n$. Let $r = 4n \ln \ln n / \ln n$. Let moreover \overline{K}_r and $\overline{K}_{r,r}$ be the complements of the complete graph on r vertices and the complete bipartite graph with each set of bipartition of cardinality r . Denote by X_r and $X_{r,r}$ the number of \overline{K}_r and $\overline{K}_{r,r}$ in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$, respectively. Then

$$\begin{aligned} \mathbb{E}X_r &= \binom{n}{r} (1 - \hat{p}_2)^{\binom{r}{2}} (1 - \hat{p}_3)^{\binom{r}{3} + \binom{r}{2}(n-r)} \\ &\leq \left(\frac{en}{r} \exp\left(-\frac{r}{2}(\hat{p}_2 + (n - r)\hat{p}_3) + o(1)\right)\right)^r \\ &\leq \left(\frac{e \ln n}{4 \ln \ln n} \exp(-2\theta_n \ln \ln n)\right)^r = o(1) \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}X_{r,r} &\leq \binom{n}{r}^2 (1 - \hat{p}_2)^{r^2} (1 - \hat{p}_3)^{2\binom{r}{2}r + r^2(n-2r)} \\ &\leq \left(\frac{en}{r} \exp\left(-\frac{r}{2}(\hat{p}_2 + (n - r)\hat{p}_3) + o(1)\right)\right)^{2r} = o(1). \end{aligned}$$

Therefore with high probability $X_r = 0$ which implies that with high probability

$$\alpha(G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)) \leq r.$$

Moreover with high probability $X_{r,r} = 0$ thus with high probability for any $S \subseteq \mathcal{V}$ such that $r \leq |S|$ we have

$$N_3(S) \geq n - |S| - r.$$

(Otherwise $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ would contain $\overline{K}_{r,r}$.)

Therefore with high probability for any $S \subseteq \mathcal{V}$ such that $r \leq |S| \leq 2n/3$

$$N_3(S) \geq \min\{|S|, r\} \geq \min\{|S|, \alpha(G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3))\}.$$

Moreover with high probability for any $S \subseteq \mathcal{V}$ such that $r \leq |S| \leq n/4$ we have

$$N_3(S) \geq n - |S| - r \geq 2|S|.$$

This finishes the proof of (i) and (ii).

(iii) For any two disjoint sets $S \subseteq \mathcal{V}$ and $S' \subseteq \mathcal{V}$ we denote by $e(S; S')$ the number of edges in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ with one end in S and one end in S' and by $e(S)$ the number of edges in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ with both ends in S . We will bound numbers $e(S)$ and $e(S, S')$ for $|S| \leq n^\gamma$ and $S' \subseteq \mathcal{V} \setminus S$ such that $|S'| = O(|S|)$. Let $S \subseteq \mathcal{V}$, $S' \subseteq \mathcal{V} \setminus S$, $|S| = s$, $1 \leq s \leq n^\gamma$ and $|S'| = 2ks$. A pair $\{v, v'\}$ ($v \in S$ and $v' \in S'$) is an edge in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ if one of the three following events occurs: $\{v, v'\}$ is an edge in $G_2(n, \hat{p}_2)$, there is a hyperedge $\{v, v', v''\}$ with $v'' \in \mathcal{V} \setminus (S \cup S')$ in $H_3(\hat{p}_3)$, there is a hyperedge $\{v, v', v''\}$ with $v'' \in S \cup S'$ in $H_3(\hat{p}_3)$. At least one of the two first events occur with probability at most $\hat{p}_2 + (n - (2k+1)s)\hat{p}_3 = O(\ln n/n)$ independently for all $v \in S$ and $v' \in S'$. Moreover each hyperedge with all vertices in $S \cup S'$ appears independently with probability $\hat{p}_3 = O(\ln n/n^2)$ and generates two edges between S and S' . Therefore

$$\begin{aligned} & \Pr \left\{ \exists_{\substack{S, S' \subseteq \mathcal{V}, \\ |S'| = 2k|S| \\ |S| \leq n^\gamma}} e(S, S') \geq 4(k+1)|S| \right\} \\ & \leq \sum_{s=1}^{n^\gamma} \binom{n}{s} \binom{n-s}{2ks} \left[\binom{2ks^2}{(2k+2)s} (\hat{p}_2 + (n - (2k+1)s)\hat{p}_3)^{(2k+2)s} + \binom{\binom{s}{2}2ks + \binom{2ks}{2}s}{(k+1)s} \hat{p}_3^{(k+1)s} \right] \\ & \leq \sum_{s=1}^{n^\gamma} \left(O(1) \left(\frac{n}{s} \right)^{2k+1} \left(\frac{s \ln n}{n} \right)^{2k+2} \right)^s + \left(O(1) \left(\frac{n}{s} \right)^{2k+1} s^{2(k+1)} \left(\frac{\ln n}{n^2} \right)^{k+1} \right)^s \\ & \leq \sum_{s=1}^{n^\gamma} \left(O(1) \frac{\ln^{2k+2} n}{n^{1-\gamma}} \right)^s + \left(O(1) \frac{\ln^{k+1} n}{n^{1-\gamma}} \right)^s \\ & = o(1). \end{aligned}$$

Similarly

$$\begin{aligned}
& \Pr \left\{ \exists_{\substack{S \subseteq \mathcal{V} \\ |S| \leq n^\gamma}} e(S) \geq 5|S| \right\} \\
& \leq \sum_{s=1}^{n^\gamma} \binom{n}{s} \left[\binom{s}{2s} (\hat{p}_2 + (n-s)\hat{p}_3)^{2s} + \binom{s}{s} \hat{p}_3^s \right] \\
& \leq \sum_{s=1}^{n^\gamma} \left(O(1) \frac{\ln^2 n}{n^{1-\gamma}} \right)^s + \left(O(1) \frac{\ln n}{n^{1-\gamma}} \right)^s \\
& = o(1).
\end{aligned}$$

Therefore with high probability for any set S ($1 \leq |S| \leq n^\gamma$) of vertices of degree at least $4k + 15$ in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ we have

$$|N_3(S)| \geq 2k|S|.$$

Otherwise for $S' = N_3(S)$ there would be $|S'| = |N_3(S)| \leq 2k$ and $e(S, N_3(S)) + 2e(S)$ would exceed $(4(k+1) + 10)|S|$.

(iv) The probability that in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ there are two vertices of degree at most C at distance at most 5 is at most

$$\begin{aligned}
& n^2 \sum_{l=0}^4 n^l \\
& \cdot (\hat{p}_2 + n\hat{p}_3)^{l+1} \left(\sum_{i=0}^C \sum_{j=0}^{\lfloor i/2 \rfloor} \binom{n-1-l}{j} \hat{p}_3^j \binom{n-1-l}{i-2j} \hat{p}_2^{i-2j} (1-\hat{p}_2)^{n-1-i-l} (1-\hat{p}_3)^{\binom{n-1-l}{2} - \binom{i+l}{2}} \right)^2 \\
& = O(1)n(\ln n)^C e^{-2n(\hat{p}_2 + \frac{n}{2}\hat{p}_3)} = o(1).
\end{aligned}$$

(v) Follows by Theorem 8.1 from [4]. □

Lemma 5. *Let k be a positive integer*

$$\hat{p}_2 + \frac{n}{2}\hat{p}_3 = \frac{\ln n + (k-1) \ln \ln n + c_n}{n}.$$

(i) *If $k = 1$ then*

$$\Pr \{ \delta(G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)) \geq 1 \} \rightarrow f(c_n).$$

(ii) *If $c_n \rightarrow \infty$ then*

$$\Pr \{ \delta(G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)) \geq k \} \rightarrow 1.$$

Proof. (i) Follows by Theorem 3.10 from [16].

(ii) Let X_t be the number of vertices of degree $0 \leq t \leq k-1$ in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$. Then

$$\begin{aligned}
\mathbb{E}X_t & \leq n \binom{n-1}{t} \sum_{i=0}^t \binom{t}{i} \hat{p}_3^i \hat{p}_2^{\max\{t-2i, 0\}} (1-\hat{p}_2)^{n-t-1} (1-\hat{p}_3)^{\binom{n-1}{2} - \binom{t}{2}} \\
& = O(1)n(\ln n)^t \exp(-\ln n + (k-1) \ln \ln n + c_n) = o(1).
\end{aligned}$$

Thus with high probability $X_t = 0$ for all $t \leq k-1$. □

Lemma 6. *Let k be a positive integer, $\theta_n \rightarrow \theta$, $\theta'_n \rightarrow \theta'$ and $\theta, \theta' > \frac{1}{2}$. Let $\mathbb{G}(n)$ be a random graph such that for*

$$(21) \quad \hat{p}_2 + \frac{n}{2}\hat{p}_3 = \frac{\theta_n \ln n}{n} \text{ and } \hat{p}_2 = \frac{\theta'_n \ln n}{n}$$

we have

$$(22) \quad G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3) \preceq_{1-o(1)} \mathbb{G}(n)$$

and in the probability space of the coupling with high probability all vertices of degree at most $4k+14$ in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ are at distance at least 6 in $\mathbb{G}(n)$. If $\Pr\{\delta(\mathbb{G}(n)) \geq k\}$ is bounded away from zero by a constant then

$$(23) \quad \Pr\{\mathbb{G}(n) \in \mathcal{C}_k | \delta(\mathbb{G}(n)) \geq k\} \rightarrow 1,$$

$$(24) \quad \Pr\{\mathbb{G}(2n) \in \mathcal{PM} | \delta(\mathbb{G}(2n)) \geq 1\} \rightarrow 1.$$

In particular, if we substitute $\mathbb{G}(n) = G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ then by Lemmas 5 and 6 we obtain the following result.

Lemma 7. *Let \hat{p}_2 and \hat{p}_3 fulfil (21).*

If $\hat{p}_2 + \frac{n}{2}\hat{p}_3 = (\ln n + c_n)/n$ then

$$\Pr\{G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3) \in \mathcal{C}_1\} \rightarrow f(c_n);$$

$$\Pr\{G_2(2n, \hat{p}_2(2n)) \cup G_3(2n, \hat{p}_3(2n)) \in \mathcal{PM}\} \rightarrow f(c_{2n}),$$

where $f(\cdot)$ is defined by (5).

If $\hat{p}_2 + \frac{n}{2}\hat{p}_3 = (\ln n + (k-1)\ln \ln n + c_n)/n$ and $c_n \rightarrow \infty$ then

$$\Pr\{G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3) \in \mathcal{C}_k\} \rightarrow 1.$$

Proof of Lemma 6. Denote by $\mathbb{G}(n)_{\delta \geq k}$ a graph $\mathbb{G}(n)$ under condition $\delta(\mathbb{G}(n)) \geq k$. If $\Pr\{\delta(\mathbb{G}(n)) \geq k\}$ is bounded away from zero and $\mathbb{G}(n)$ has certain property with high probability, then also $\mathbb{G}(n)_{\delta \geq k}$ has this property with high probability.

From now on we assume that $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ and $\mathbb{G}(n)$ are defined on the same probability space existing by (22). We call a vertex $v \in \mathcal{V}$ small if its degree in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ is at most $4k+14$. Otherwise we call a vertex large. Let $S \subseteq \mathcal{V}$ and $|S| \leq n^\gamma$. Denote by S^+ and S^- the subset of large and small vertices of S , respectively. Then by Lemma 4(iii) with high probability

$$|N_{\mathbb{G}(n)_{\delta \geq k}}(S^+)| \geq |N_3(S^+)| \geq 2k|S^+|.$$

Moreover in $\mathbb{G}(n)_{\delta \geq k}$ all vertices in S^- have degree at least k and with high probability are at distance at least 6. Therefore with high probability in $\mathbb{G}(n)_{\delta \geq k}$ no two vertices in S^- are connected by an edge or have a common neighbour and at most $|S^+|$ of them have neighbours in $N_3(S^+) \cup S^+$ (otherwise they would be connected by a path of length at most 5). Thus with high probability

$$\forall_{S \subseteq \mathcal{V}, 1 \leq |S| \leq n^\gamma} |N_{\mathbb{G}(n)_{\delta \geq k}}(S)| \geq |N_{\mathbb{G}(n)_{\delta \geq k}}(S^+)| + k \max\{|S^-| - |S^+|, 0\} \geq k|S|.$$

If we combine this with Lemma 4(i) and (ii) we get that with high probability

(25)

$$N_{\mathbb{G}(n)_{\delta \geq k}}(S) \geq k \quad \text{for all } S \subseteq \mathcal{V}, 1 \leq |S| \leq \frac{n}{2};$$

(26)

$$N_{\mathbb{G}(n)_{\delta \geq k}}(S) \geq \min\{|S|, 4n \ln \ln n / \ln n\} \geq \min\{|S|, \alpha(\mathbb{G}(n)_{\delta \geq k})\} \quad \text{for all } S \subseteq \mathcal{V}, 1 \leq |S| \leq \frac{2n}{3};$$

(27)

$$N_{\mathbb{G}(n)_{\delta \geq k}}(S) \geq 2|S| \quad \text{for all } S \subseteq \mathcal{V}, 1 \leq |S| \leq \frac{n}{4}.$$

Finally (23) follows immediately by (25). Moreover if (26) is fulfilled then $\mathbb{G}(n)_{\delta \geq k}$ has a perfect matching (for the proof see for example [3]). Therefore (24) follows. (27) will be used later to establish threshold function for a Hamilton cycle. \square

Lemma 8. *Let k be a positive integer, $\theta_n \rightarrow \theta$, $\theta'_n \rightarrow \theta'$, $\theta, \theta' > \frac{1}{2}$ and*

$$\hat{p}_2 + \frac{n}{2}\hat{p}_3 = \frac{\theta_n \ln n}{n}, \quad \hat{p}_2 = \frac{\theta'_n \ln n}{n} \quad \text{and} \quad \hat{p}_4 = \frac{512}{n}.$$

Let moreover $\mathbb{G}(n)$ be a random graph such that:

- (i) $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3) \preceq_{1-o(1)} \mathbb{G}(n)$;
- (ii) *with high probability $\delta(\mathbb{G}(n)) \geq 2$;*
- (iii) *in a probability space existing by (i) all vertices of degree at most 22 in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ are at distance at least 6 in $\mathbb{G}(n)$.*

Then

$$\Pr \{ \mathbb{G}(n) \cup G_2(n, \hat{p}_4) \in \mathcal{HC} \} \rightarrow 1.$$

In particular if $\hat{p}_2 + \frac{n}{2}\hat{p}_3 = (\ln n + \ln \ln n + c_n)/n$ and $c_n \rightarrow \infty$ then

$$\Pr \{ G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3) \in \mathcal{HC} \} \rightarrow 1.$$

Proof. We will follow the lines of the proof of Theorem 8.9 from [4]. Let $t = 8n/\ln n$ and $\hat{p}_{4,0} = 64 \ln n / n^2$. Then $t\hat{p}_{4,0} = \hat{p}_4$. For any graph G let $l(G)$ be the length of the longest path and $l(G) = n$ if G has a Hamilton cycle. We say that G has property \mathcal{Q} if

$$G \text{ is connected} \quad \text{and} \quad |N_G(S)| \geq 2|S|, \text{ for all } S \subseteq \mathcal{V}, |S| \leq n/4.$$

In the proof we assume that $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ and $\mathbb{G}(n)$ are defined on the probability space of the coupling existing by (i). Let $\mathbb{G}_0 = \mathbb{G}(n)$ and

$$\mathbb{G}_i = \mathbb{G}_{i-1} \cup G_2(n, \hat{p}_{4,0}), \quad \text{for } 1 \leq i \leq t.$$

By Lemma 6, assumptions (i)–(iii), and (27) with high probability $\mathbb{G}(n)$ has property \mathcal{Q} . Moreover by Lemma 4(v) with high probability $l(\mathbb{G}_0) \geq 1 - (8 \ln 2 / \ln n)$. It is shown in [4] that

$$\Pr \{ l(\mathbb{G}_i) = n - t + i - 1 | l(\mathbb{G}_{i-1}) = n - t + i - 1 \text{ and } \mathbb{G}_{i-1} \text{ has } \mathcal{Q} \} \leq (1 - \hat{p}_{4,0})^{n^2/32} \leq n^{-2}.$$

Thus

$$\Pr \{l(\mathbb{G}_t) = n\} \geq 1 - \frac{t}{n^2} - o(1) = 1 - o(1).$$

Since

$$\mathbb{G}_t \preceq \mathbb{G}(n) \cup G_2(n, t\hat{p}_{4,0}),$$

this finishes the proof. \square

6 Sharp thresholds

Proof of Theorems 2–4. First let $S_1 = n(\ln n + c_n)$ and \hat{p}_2 and \hat{p}_3 be given by (2). Note that $\hat{p}_2 > \frac{1+\varepsilon}{2} \ln n$ for some constant $\varepsilon > 0$. Then by Theorem 1 and Lemma 7

$$\lim_{n \rightarrow \infty} \Pr \{\mathcal{G}(n, m, \bar{p}) \in \mathcal{C}_1\} \geq \lim_{n \rightarrow \infty} \Pr \{G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3) \in \mathcal{C}_1\} = f(c_n)$$

and by Lemma 1

$$\lim_{n \rightarrow \infty} \Pr \{\mathcal{G}(n, m, \bar{p}) \in \mathcal{C}_1\} \leq \lim_{n \rightarrow \infty} \Pr \{\delta(\mathcal{G}(n, m, \bar{p})) \geq 1\} = f(c_n).$$

This implies Theorem 2(i). Remaining theorems follow by Theorem 1 and the following results: Theorem 2(ii) by Lemma 7 and Lemma 2; Theorem 3 by Lemma 7 and Lemma 1; Theorem 4 by Lemma 8 and Lemma 2. \square

In the following proofs we will assume that $c_n = O(\ln n)$. In the other cases theorems follow by Lemma 1 or (3) combined with known results concerning $G_2(n, \hat{p})$.

Proof of Theorem 5. Let $c_n \rightarrow -\infty$, then by Lemma 3

$$\Pr \{\mathcal{G}(n, m, p) \in \mathcal{HC}\} \leq \Pr \{\delta(\mathcal{G}(n, m, p)) \geq 2\} \rightarrow 0.$$

Let now $c_n \rightarrow \infty$. First consider the case $m = \Omega(n/\ln n)$. Let moreover

$$m' = m \left(1 + \frac{700}{\ln n}\right)^{-1} \quad \text{and} \quad m'' = \frac{700}{\ln n} m'.$$

Then for p given by (6)

$$m'p(1 - (1 - p)^{n-1}) = \ln n + \ln \max \left\{1, \frac{np e^{-np} \ln n}{1 - e^{-np}}\right\} - 700 + o(1) + c_n$$

and

$$\begin{aligned} nm'p^2 &= O(\ln n) && \text{for } np = O(1), \\ nm'p^2 &= \frac{n}{m'}(m'p)^2 = O(\ln^3 n) && \text{for } np \rightarrow \infty. \end{aligned}$$

Let $\omega \rightarrow \infty$. Define

$$\begin{aligned}\hat{p}_2 &= \begin{cases} \frac{m'p(1-(1-p)^{n-1}) - 3m'p\left(\frac{1-(1-2p)^n}{2np} - (1-p)^{n-1}\right) - \omega\sqrt{\frac{\ln n}{n}}}{n}, & \text{for } nm'p\left(\frac{1-(1-2p)^n}{2np} - (1-p)^{n-1}\right) \gg \sqrt{n \ln n}; \\ \frac{m'p(1-(1-p)^{n-1}) - \omega\sqrt{\frac{\ln n}{n}}}{n}, & \text{otherwise;} \end{cases} \\ \hat{p}_3 &= \begin{cases} \frac{m'p\left(\frac{1-(1-2p)^n}{2np} - (1-p)^{n-1}\right) - \omega\sqrt{\frac{\ln n}{n}}}{\frac{n^2}{6}}, & \text{for } nm'p\left(\frac{1-(1-2p)^n}{2np} - (1-p)^{n-1}\right) \gg \sqrt{n \ln n}; \\ 0, & \text{otherwise;} \end{cases} \\ \hat{p}_4 &= \frac{m''p\left(1 - \frac{1-(1-2p)^n}{2np}\right)}{n} \geq \frac{512}{n}.\end{aligned}$$

Let now $\mathbb{G}(n) = \mathcal{G}(n, m', p)$. Then by Theorem 1

$$(28) \quad \begin{aligned} G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3) &\preceq_{1-o(1)} \mathcal{G}(n, m', p) = \mathbb{G}(n); \\ G_2(n, \hat{p}_4) &\preceq_{1-o(1)} \mathcal{G}(n, m'', p). \end{aligned}$$

Moreover by Lemma 3 with high probability $\delta(\mathbb{G}(n)) \geq 2$. Therefore assumptions (i) and (ii) in Lemma 8 are fulfilled.

We are left with proving (iii). Let C be a positive integer. We will show that in the probability space of the coupling (28) with high probability any two vertices of degree at most C in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ are at distance at least 6 in $\mathcal{G}(n, m', p) = \mathbb{G}(n)$. Therefore we need to study the couplings described in the proof of Theorem 1. Recall that

$$\begin{aligned} G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3) &= \mathbb{G}_{*2}\left(n, \text{Po}\left(\binom{n}{2} \ln(1 - \hat{p}_2)^{-1}\right)\right) \cup \mathbb{G}_{*3}\left(n, \text{Po}\left(\binom{n}{3} \ln(1 - \hat{p}_3)^{-1}\right)\right) \\ &\preceq^{(i)} \mathbb{G}_{*2}\left(n, \text{Po}\left(\frac{S_1 - 3S_3 - 5\omega'\sqrt{S_1}}{2}\right)\right) \cup \mathbb{G}_{*3}\left(n, \text{Po}\left(S_3 - 2\omega'\sqrt{S_1}\right)\right) \\ &\preceq_{1-o(1)}^{(ii)} \mathbb{G}_{*2}\left(n, \frac{S_1 - 3S_3 - 4\omega'\sqrt{S_1}}{2}\right) \cup \mathbb{G}_{*3}\left(n, S_3 - \omega'\sqrt{S_1}\right) \\ &\preceq_{1-o(1)}^{(iii)} \mathbb{G}_{*2}(n, M_2) \cup \mathbb{G}_{*3}(n, M_3) = \bigcup_{1 \leq i \leq m'} (\mathbb{G}_{*2}(n, \frac{Y_i - 3Z_i}{2}) \cup \mathbb{G}_{*3}(n, Z_i)) \\ &\preceq^{(iv)} \mathcal{G}(n, m', p) \end{aligned}$$

for some $\omega' = \Theta(\omega)$.

First we will show that in the probability space of the above couplings with high probability the degree of a vertex $v \in \mathcal{V}$ in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ is well approximated by the number of those features contained in $W(v)$, which are chosen by at least two vertices (i.e. which contribute to at least one edge in $\mathcal{G}(n, m', p)$). For any $v \in \mathcal{V}$ in $\mathcal{G}(n, m', p)$ let

$$W'(v) = \{w_i \in W(v) : |V'_i| \geq 2\},$$

where V'_i is defined in (9). Denote by \mathcal{A}_v event that while constructing $\mathbb{G}_{*2}(n, M_2) \cup \mathbb{G}_{*3}(n, M_3)$ in less than $|W'(v)| - 1$ draws we choose an edge or hyperedge containing

v . Given $v \in \mathcal{V}$ and $1 \leq i \leq m'$. Denote by $\mathcal{A}_{v,i,1}$ event that v is an isolated vertex in $\mathbb{G}_{*2}(n, (Y_i - 3Z_i)/2) \cup \mathbb{G}_{*3}(n, Z_i)$ and $\mathcal{A}_{v,i,2}$ event that $v \in V'_i$. Then

$$\mathcal{A}_v \subseteq \bigcup_{i \neq j} (\mathcal{A}_{v,i,1} \cap \mathcal{A}_{v,i,2}) \cap (\mathcal{A}_{v,j,1} \cap \mathcal{A}_{v,j,2}).$$

For $i \neq j$, $(\mathcal{A}_{v,i,1} \cap \mathcal{A}_{v,i,2})$ and $(\mathcal{A}_{v,j,1} \cap \mathcal{A}_{v,j,2})$ are independent and $\Pr\{\mathcal{A}_{v,i,1} \cup \mathcal{A}_{v,i,2}\} = 1$, thus

$$\begin{aligned} \Pr\{\mathcal{A}_v\} &\leq \binom{m'}{2} (\Pr\{\mathcal{A}_{v,1,1} \cap \mathcal{A}_{v,1,2}\})^2 \\ &= \binom{m'}{2} (\Pr\{\mathcal{A}_{v,1,1}\} + \Pr\{\mathcal{A}_{v,1,2}\} - 1)^2 \\ &\leq \binom{m'}{2} \left((1-p)^n + np(1-p)^{n-1} + \sum_{y=2}^{n-1} \binom{n}{y} p^y (1-p)^{n-y} \left(1 - \frac{1}{n}\right)^y + p^n \cdot 0 \right. \\ &\quad \left. + \sum_{y=2}^n \binom{n}{y} p^y (1-p)^{n-y} \frac{y}{n} - 1 \right)^2 \\ &= \binom{m'}{2} \left(\sum_{y=0}^n \binom{n}{y} p^y (1-p)^{n-y} \left(1 - \frac{1}{n}\right)^y - p^n + \sum_{y=1}^n \binom{n}{y} p^y (1-p)^{n-y} \frac{y}{n} - 1 \right)^2 \\ &\leq \binom{m'}{2} \left(\left(1 - \frac{p}{n}\right)^n + p - 1 \right)^2 \leq (m')^2 p^4 \end{aligned}$$

Therefore

$$\Pr\left\{\bigcup_{v \in \mathcal{V}} \mathcal{A}_v\right\} \leq n(m')^2 p^4 = O\left(\frac{\ln^6 n}{n}\right) = o(1).$$

Recall that random graph $\mathbb{G}_{*2}(n, \cdot) \cup \mathbb{G}_{*3}(n, \cdot)$ is constructed by making independent draws of edges and hyperedges in an auxiliary hypergraph. The number of draws is given by random variables. Couplings (i)–(iii) may rely on this construction. In the coupling (i)–(iii) in order to get from $\mathbb{G}_{*2}(n, \text{Po}(\binom{n}{2} \ln(1 - \hat{p}_2)^{-1})) \cup \mathbb{G}_{*3}(n, \text{Po}(\binom{n}{3} \ln(1 - \hat{p}_3)^{-1}))$ a graph $\mathbb{G}_{*2}(n, M_2) \cup \mathbb{G}_{*3}(n, M_3)$ with high probability we make some additional draws. Moreover by the sharp concentration of the Poisson distribution and (12) with high probability the number of additional draws is at most $K\omega\sqrt{n \ln n}$ for some constant K . Under condition that the number of additional draws is bounded at most $K\omega\sqrt{n \ln n}$, probability that an edge (hyperedge) containing v is chosen in at least 2 additional draws is at most.

$$\binom{K\omega\sqrt{n \ln n}}{2} \left(\frac{2}{n}\right)^2 = o(1).$$

Moreover each draw (each chosen hyperedge) generates at most 2 edges incident to v . Concluding in the probability space of the coupling (28) with high probability the number of edges incident to v in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ is at least $W'(v) - 2 - 2$. Therefore if degree of v in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ is at most C then $W'(v) \leq C + 4$.

Finally, probability that there are two vertices v, v' such that $|W'(v)| \leq C + 4$ and $|W'(v')| \leq C + 4$ connected by a path of length at most 5 in $\mathcal{G}(n, m', p)$ is at most

$$n^2 \sum_{t=1}^5 (m')^t n^{t-1} p^{2t} \left(\sum_{l=0}^{C+4} \binom{m'-t}{l} (p - p(1-p)^{n-1})^l [(1-p) + p(1-p)^{n-1}]^{m'-t-2C-4} \right)^2 \\ = O(1)n(\ln n)^{O(1)} \exp(-2m'p(1 - (1-p)^{n-1})) = o(1).$$

Therefore by Lemma 8 with high probability

$$\mathbb{G}(n) \cup G_2 \left(n, \frac{512}{n} \right) \in \mathcal{HC}.$$

Thus in the case $m = \Omega(n/\ln n)$ the theorem follows by (28) and a straight forward coupling of random intersection graphs.

$$\mathbb{G}(n) \cup G_2 \left(n, \frac{512}{n} \right) \preceq_{1-o(1)} \mathcal{G}(n, m', p) \cup \mathcal{G}(n, m'', p) \preceq \mathcal{G}(n, m, p)$$

Consider now the case $m = o(n/\ln n)$. Note that in this case (6) implies

$$p = \frac{\ln n + c'_n}{m}$$

for some $c'_n \rightarrow \infty$. Let m' be such that m divides m' and $m' \sim n/\ln n$. Take now an instance of $\mathcal{G}(n, m', p')$ with the set of features \mathcal{W}' of size m' and

$$p' = \frac{\ln n + c'_n}{m'}.$$

Divide \mathcal{W}' into m groups of m'/m features. Denote by S_i the set of vertices which have chosen features from the i -th, $1 \leq i \leq m$, group in $\mathcal{G}(n, m', p')$. $|S_i|$ has binomial distribution $\text{Bin}(n, p'')$ for some $p'' \leq \frac{m'}{m} p' = p$. Now take an instance of $\mathcal{G}(n, m', p')$ and construct the sets V_i , $1 \leq i \leq m$, in $\mathcal{G}(n, m, p)$ by taking S_i and adding to V_i each vertex from $\mathcal{V} \setminus S_i$ independently with probability $(p - p'')/(1 - p'')$. This coupling implies

$$\mathcal{G}(n, m', p') \preceq \mathcal{G}(n, m, p).$$

From the considerations concerning the first case we have that with high probability $\mathcal{G}(n, m', p') \in \mathcal{HC}$. Therefore also with high probability $\mathcal{G}(n, m, p) \in \mathcal{HC}$. \square

Proof of Theorem 6. The technique of the proof is analogous to this of the proof of Theorem 5. We consider two cases $m = \Omega(n/\ln n)$ and $m = o(n/\ln n)$. In the first case the proof relies on Lemma 3 and Lemma 6. Moreover we have to use the fact shown in the proof of Theorem 5, that with high probability in the coupling constructed to prove Theorem 1 with high probability the vertices of degree bounded by a constant in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ are at distance at least 6 in $\mathcal{G}(n, m, p)$. The proof of the case $m = o(n/\ln n)$ is the same as in the proof of Theorem 5. \square

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